THE MATHEMATICS TEACHER

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AN EXPERIMENT IN GRADING PROBLEMS IN ALGEBRA.

By Edward L. Thorndike.

The Difficulty of Certain Problems in Algebra as Judged by the Consensus of Two Hundred Teachers of Mathematics.

Two hundred teachers of mathematics, chiefly members of the New York Section of the Association of Teachers of Mathematics of the Middle States and Maryland, ranked the twenty-five problems printed below for difficulty, "difficulty" being defined as in the instructions appended. The variations in the individual opinions were very great, being as shown in Table I. It is an interesting exercise to examine this table, and imagine, as well as one can, the points of view from which these varying estimates were each plausible—to divine, for example, why Problem *T* was rated all the way from easiest to hardest of the twenty-five. How much of the variation was due to tenable points of view and how much was due to errors of judgment cannot, of course, be told until the problem in question has been tested with respect to the percentage of pupils able to solve it in the time allowed.

In spite of the great variation, the consensus of the two hundred teachers gives a fairly clear order of difficulty for the problems, as shown at the bottom of Table I. *D* is easiest; *K*

TABLE I.

THE RELATIVE POSITIONS ASSIGNED TO EACH OF 25 PROBLEMS IN ALGEBRA BY 200 TEACHERS OF MATHEMATICS.

The entries in the body of the table give the frequencies of each problem in each position. The first line below the body of the table gives the position or point between two positions which most nearly divides the ratings into halves. The next three lines give the results of the division. The last line gives the most probably true order of difficulty by the consensus.

Position	+D	K	A	L	O	S	T	P	U	G	B	C	Q	H	R	E	M	I	X	Y	J	N	V	F	W	
1	146	32	7	8			3			1	3								1							1
2	43	121	19	10	2		3		1																	2
3	8	21	75	25	52	7	3			3	4				1				1							3
4	1	17	27	65	26	41	8	2	5	3	3								1			1				4
5	1	4	26	26	63	19	21	12	11	7	7	2														5
6	1	3	16	23	26	61	16	11	14	8	12	1	2		3				2		1					6
7		1	16	11	17	31	24	32	15	27	10	4	3		7		2	1								7
8		1	5	17	10	15	22	45	28	18	18	7	8	1	3	1		3	1		1					8
9			5	8	2	10	14	43	33	24	16	20	4	5	12	1					2					9
10			2	4		8	17	25	27	20	24	27	13	13	11	2			1	1		3	1	1	1	10
11			1	2		3	16	14	16	21	16	47	14	20	14	6	5	2	1		1					11
12				1		1	11	8	15	12	14	33	29	29	16	13	7	5	5				3			12
13					2	1	12	6	20	9	9	24	36	33	9	15	14	3	2	3	1	2				13
14						3	7	1	11	7	13	19	25	39	11	25	19	3	9		5		3	1		14
15			1				4	1	2	8	11	7	23	32	16	35	25	6	9	7	5	3	5		1	15
16							5			9	6	6	19	14	10	30	34	11	13	12	19	6	1	4		16
17							6		1	3	13	1	10	8	15	24	33	16	14	19	20	11	4	1		17
18							2			5	12	2	3	4	16	20	30	20	19	24	15	13	10	4	2	18
19							2		1	6	2		8	1	11	13	17	31	22	31	19	15	13	3	2	19
20							1			1	2		1		9	7	4	28	32	26	24	24	21	12	7	20
21							1			2	5		1	1	11	4	8	23	22	30	11	27	21	21	13	21
22										1					11	1	1	22	14	19	19	39	43	17	14	22
23							1			1					7	1	1	12	12	15	15	26	43	43	22	23
24										2			1		5	1		8	17	8	28	21	20	50	38	24
25							1			2					2	1		6	2	5	14	9	12	43	100	25
Approximate Median Position	1	2	3–4	4–5	5	6	8–9	8–9	9–10	10	10–11	11–12	13–14	13–14	15–16	15–16	16–17	19–20	19–20	19–20	20	21–22	22	23–24	24–25	
Data on median position		32 121 47	101 99	108 92	80 63 57	67 61 72	100 100	102 98	107 93	91 20 89	97 103	108 92	109 91	101 99	103 97	98 102	106 94	101 99	101 99	97 103	89 24 87	105 95	82 43 75	107 93	100 100	

is next easiest; *A* is next; and so on (*T* and *P* being closely alike and *I*, *X* and *Y* being closely alike). These position-values are, however, of comparatively little use. The significance of any one of them depends upon consideration of the entire series, and is, as it stands, only a means of relative position, not of the amount of "difficulty" or of anything else.

List of Problems.

To be ranked in order of difficulty. See sheet of instructions.

A. If $x + 3a = 5a$, what does x equal?

B. The circumference of a circle $= 2\pi r$. $\pi = 3\frac{1}{7}$. $r =$ the length of the radius of the circle in question. If the diameter of a bicycle wheel is 28 inches, how many inches is the circumference?

C. If $\frac{6x+7}{5} - \frac{2x-1}{10} = 4\frac{1}{2}$, what does x equal?

D. If $a = 4$ and $b = 2$, what does $a + b$ equal?

E. If $2 + \dfrac{\frac{x}{a} - 1}{\frac{2}{a}} = 0$, what does x equal?

F. A cube containing eight cubic inches was plated with copper. The difference in the weights of the cube before and after the plating was 0.139 lbs. 1 cubic inch of copper weighs 0.315 lbs. Form an equation from which the approximate thickness of the copper plating could be calculated. State whether the approximate estimated thickness by your equation would be less or more than the exact thickness.

G. If $a = 6$ and $b = 3$, what does $\sqrt{a}\,\sqrt{2b}$ equal?

H. If $\frac{1}{a} - \frac{1}{x} = \frac{1}{x} - \frac{1}{b}$, what does x equal?

I. A man has a hours to spend riding with a friend. How far can they ride together, going out at the rate of b miles an hour, and just covering the return trip at the rate of c miles an hour?

J. If $\frac{a+b}{b+c}=\frac{c+d}{d+a}$, prove that $a=c$ or that $a+b+c+d=0$.

K. If $a=4$ and $b=0$, what does $a+b$ equal?

L. If $3x+4=2x+8$, what does x equal?

M. If $\frac{x+a}{x-a}-\frac{x-a}{x+a}-\frac{x^2}{a^2-x^2}=1$, what does x equal?

N. There are two thermometers or scales to measure temperature. The Fahrenheit scale (F.) is the one we commonly use. The other is called the Centigrade scale (C.). A temperature of 32 degrees on the F. scale = 0 degrees on the C. scale. 33.8 degrees on the F. scale = 1 degree on the C. scale. 35.6 degrees on F. = 2 degrees on C. 50 degrees on F. = 10 degrees on C. 14 degrees on F. = — 10 degrees on C.

(*a*) What on the C. scale = 70 on the F. scale.
(*b*) What on the C. scale = 4 degrees *below* zero on the F. scale?
(*c*) What on the F. scale = 20 degrees on the C. scale?

O. If $a=3$ and $b=2$, what does a^2-ab equal?

P. If $x-2a+b=2x+2b-4a$, what does x equal?

Q. If $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}$, what does x equal?

R. Let l stand for the safe load that can be hoisted by a hemp rope. Let c stand for the circumference of a rope. If $l=100c^2$ for any hemp rope, how many pounds are a safe load for a hemp rope $2\frac{1}{4}$ inches in circumference?

S. If $a=6$ and $b=1$, what does $2ab-ab^2$ equal?

T. Find the average midnight temperature for the week in which the daily midnight temperatures were 15, 3, 0, —7, —9, 6, and 17 degrees.

U. If $\frac{x}{a+b}=a-b$, what does x equal?

V. How much water must be added to a pint of "alcohol, 95% pure," to make a solution of "alcohol, 40% pure"?

W. Given that $2x - 3$ is less than $x + 5$ and that $11 + 2x$ is less than $3x + 5$, to find the limits (*i. e.*, the values) between which x lies.

(It is understood that the pupil has not had any special training in inequalities or limits. This problem is, so to speak, an original exercise.)

X. At what time between 6 and 6:30 o'clock are the hands of a watch at right angles to each other?

Y. If $x = \frac{a + b}{2}$ what does $\left(\frac{x - a}{x - b}\right)^3 - \frac{x - 2a + b}{x + a - 2b}$ equal?

Instructions for the Experiment.

Cut up the sheet of problems so as to have twenty-five strips, each strip with one problem. Examine the problems with a view to ranking them in order of difficulty for pupils of age from 14 years 0 months to 15 years 11 months, who have had twenty weeks' work in algebra, five periods a week, or its equivalent. Mark the easiest one, 1; mark the next easiest one, 2; mark the next easiest, 3; continue, the hardest problem being marked 25. If two or more of the problems seem to be of equal difficulty, choose one of them at random for the easiest of them, and so on. That is, suppose that you have already picked out as to difficulty your Nos. 1, 2, 3, 4, and 5, and that you then find three problems that seem equally difficult. These are to fill ranks 6, 7 and 8. Shuffle them, take one for 6, another for 7, and another for 8, at random.

In ranking the problems let "more difficult" mean, for any example, "likely to be solved correctly *in thirty minutes* by a smaller percentage of the pupils." That is, assume that thirty minutes are allowed for each problem and try to prophesy the ranking of the problems by the number out of ten thousand pupils (of the sort stated above) who would fail with each problem. If you prophesy that 2,000 pupils will fail to get Problem *A* and 1,900 will fail on Problem *B*, then call *A* more difficult than *B*.

The amounts of difficulty ascribed to these problems by the consensus of the two hundred teachers can be inferred from the percentage of them judging each example to be harder than each other example. If, that is, eighty per cent. of them judged *K* to be harder than *D*, while 90 per cent. of them judged *A* to be harder than *D*, and 99 or 100 per cent. of them judged *O* to be harder than *D*, we should (letting *d*, *k*, *a*, and *o* equal, respectively, the amounts of "difficulty" possessed by *D*, *K*, *A* and *O*

in the minds of the two hundred teachers) be confident that $o > a$, that $a > k$, and that $k > d$. If, further, eighty per cent. of them judged A to be harder than K, we may be confident that $a-k$ is approximately equal to $k-d$, *since the two differences are equally often judged correctly by the two hundred judges, there being a minority above zero.*

Counting up the 200 decisions in the case of each problem's comparison with every other, we have a table of which Table II. shows the first three lines in part as a sample. This table reads: 83 per cent. of the teachers judged that $k > d$; 93 per cent. of them judged that $a > d$; 94½ per cent. of them judged that $l > d$; 100 per cent. of them judged that $o > d$; 82 per cent. of them judged that $a > k$; 88½ per cent. of them judged that $l > k$; 97 per cent. of them judged that $o > k$; etc.

TABLE II.

THE FREQUENCY OF JUDGMENTS THAT "K IS MORE DIFFICULT THAN D," THAT "A IS MORE DIFFICULT THAN D," THAT "L IS MORE DIFFICULT THAN D," ETC., ETC. In percentages.

	K	A	L	O	S	T	P	U	G	B	C	Q	etc.
D	83	93	94½	100	100	etc.							
K		82	88½	97	97	97	etc.						
A			67	64½	71	83½	99	etc.					
L				59½	64	72½	99½	93	etc.				
O													
S													
etc.													

It is possible to use knowledge of the relation that holds good between (1) the perecentage of judges judging a certain difference correctly and (2) the amount of the difference, so as to infer the latter from the former. I will not, however, enter into either the principles or the technique of doing so for the case in hand, since a much simpler method of defining the amounts of difficulty of a series of these problems is available.

Let a, b, c, etc., equal respectively the "difficulty" of A, "difficulty" of B, etc., in the opinions of the two hundred teachers, as heretofore. Then the table from which Table II. is an excerpt would show the following:

83 % of the judges judged that $k > d$
82 % of the judges judged that $a > k$
83½% of the judges judged that $t > a$
81½% of the judges judged that $h > t$
82½% of the judges judged that $e > h$
81 % of the judges judged that $i > e$

Hence, approximately, $k-d=a-k$; $a-k=t-a$; $t-a=h-t$; $h-t=e-h$; and $e-h=i-e$.

Call the amount of difference in "difficulty" necessary in order that 82¼ per cent. of the judges should rate the harder of two problems as harder, *dif.*

Then, approximately,

the difficulty of D in the minds of the 200 judges is d
the difficulty of K in the minds of the 200 judges is $d+1$ *dif.*
the difficulty of A in the minds of the 200 judges is $d+2$ *dif.*
the difficulty of T in the minds of the 200 judges is $d+3$ *dif.*
the difficulty of H in the minds of the 200 judges is $d+4$ *dif.*
the difficulty of E in the minds of the 200 judges is $d+5$ *dif.*
the difficulty of I in the minds of the 200 judges is $d+6$ *dif.*

The difficulty of D, the difficulty of K, the difficulty of A, and so on, can now be stated as we state the weights of certain objects, or the amounts of wealth possessed by certain men, *if we can estimate the amount of difficulty of D itself*—that is, the difference between the difficulty of D and zero, or "just not any," difficulty.

A problem in algebra that approached zero difficulty might be variously defined, but perhaps the most useful definition would be: "*A problem that is truly algebraic, not simply a problem for quantitative thinking in general, and certainly not simply a problem for thought in general; and that has some difficulty, but less than any other such problem.*" For example, the problem, "Call one dime d, call two dimes $2d$, call three dimes $3d$. How many cents are there in $4d$?" is truly algebraic and would, by many competent judges, be rated as easier than D. The problem, "Call one dime d, call two dimes $2d$, call three dimes $3d$. How many cents are there in $3d$?" is also truly algebraic and is perhaps still easier. Somewhere amongst the algebraic problems that can be devised that are easier than D will be found

one (call it α) nearly at the point of "just not any" difficulty. By "nearly" is of course meant that from the true zero of difficulty to the difficulty of α is a small fraction of the difference between the true zero and, say, the difficulty of E or I.

The difference between "zero difficulty" as an algebra problem and the difficulty of D has not been determined. It is almost surely, in the opinion on the two hundred, less than 2 *dif.* and probably a little, if at all, over 1 *dif.* Call it 1 *dif.* That is, assume that zero or "just not any" difficulty as a problem in algebra is represented by a problem as much easier than D as D is easier than K.

Then

the difficulty of $D = 1$ *dif.*
the difficulty of $K = 2$ *dif.*
the difficulty of $A = 3$ *dif.*
the difficulty of $T = 4$ *dif.*
the difficulty of $H = 5$ *dif.*
the difficulty of $E = 6$ *dif.*
the difficulty of $I = 7$ *dif.*

And T is twice as difficult as K in the sense of being twice as far from zero on a scale for "difficulty" as defined by the consensus; E is one and a half times as difficult as T, twice as difficult as A; etc., etc.

By a fairly permissible hypothesis about the nature of the individual differences of these 200 teachers' powers of judging these differences in difficulty, the constant which I have here called "1 *dif.*" equals 1.371 times the amount of added difficulty which would cause 75 per cent. of these teachers to rate the harder of two problems as harder.* Since in other work with educational scales, this "*P.E.*" or "75%-difference" has been taken as a unit, we may well restate the scale above as

* An account of the nature of this hypothesis and the consequent derivation of the 1.371 P.E. is beyond the scope of this paper. Its essential feature is the assumption that the factors which make the judgments of two hundred teachers of mathematics concerning a problem's difficulty vary one from another can be arranged in *n* groups, these groups being approximately equal in magnitude of influence, and approximately uncorrelated in action, and that *n* is fairly large (say, 15 or more).

the difficulty of $D = 1.37$ P.E.
the difficulty of $K = 2.74$ P.E.
the difficulty of $A = 4.11$ P.E.
the difficulty of $T = 5.48$ P.E.
the difficulty of $H = 6.85$ P.E.
the difficulty of $E = 8.22$ P.E.
the difficulty of $I = 9.59$ P.E.

73½ per cent. of the two hundred judges judged that $v > i$; and 78½ per cent. of them, that $w > v$. For reasons which I shall here omit, we may infer that, approximately,

$$v - i = .71\ dif. \text{ or } .98 \text{ P.E.}$$

$$w - v = .85\ dif. \text{ or } 1.17 \text{ P.E.}$$

and

$$v = 7.7\ dif. \text{ or } 10.57 \text{ P.E.};\ w = 8.55\ dif. \text{ or } 11.74 \text{ P.E.}$$

So our entire series of standards, by the consensus of the two hundred judges, may be taken roughly as

d, the difficulty of D,	= 1.4 P.E.	or 1	*dif.*
k, the difficulty of K,	= 2.7 P.E.	or 2	*dif.*
a, the difficulty of A,	= 4.1 P.E.	or 3	*dif.*
t, the difficulty of T,	= 5.5 P.E.	or 4	*dif.*
h, the difficulty of H,	= 6.9 P.E.	or 5	*dif.*
e, the difficulty of E,	= 8.2 P.E.	or 6	*dif.*
i, the difficulty of I,	= 9.6 P.E.	or 7	*dif.*
v, the difficulty of V,	= 10.6 P.E.	or 7.7	*dif.*
w, the difficulty of W,	= 11.7 P.E.	or 8.5	*dif.*

These results are sound as an expression of the approximate amounts of "difficulty" of the problem in question in the opinion of the two hundred teachers in question. And such a consensus as treated gives estimates of these amounts of "difficulty" far superior to the opinion of a single teacher. Until a better scale is devised this scale may be freely used as a foot-rule to define and measure the difficulty of examples in algebra for pupils who have studied the subject for the stated length of time.

By objective tests of pupils a better scale can be devised. Its validity will depend upon the validity of the assumptions made concerning the distribution of ability to solve problems in algebra in the pupils in question. The general nature of the argument in such a case has been fully illustrated by Dr. Buckingham in the case of a scale for the spelling-difficulty of words.

It will have been understood, I trust, that the extent to which "difficulty" as judged by these two hundred teachers is identical with "difficulty" as judged by a consensus of, say, English or French, or Canadian, or Californian, teachers is a matter to be determined by experiment; and, similarly, that the extent to which any ten thousand children tested represent any other group of children is a matter to be determined by experiment. "Difficulty" is, of course, not one thing, but many, varying for the same problem with the amount of training and its nature. The scale, as given, has just the extent of application that this particular consensus warrants. Probably no one realizes the limitations of these measures of the "difficulty" of *D, K, A*, etc., so well as the author, who can easily add three to every one of the criticisms of such scales that he has received. Yet I unhesitatingly assert that, at the present time, no measures of the "difficulty" of *D, K, A*, etc., for pupils after 20 weeks' work in algebra exist that have one half the probability of being true that the measures stated here have.

Individual Differences amongst Teachers in Respect to Agreement with the Consensus in Rating 24 Problems, *G* Being Omitted from Consideration.

The order assigned to the problems by any one teacher must, Table I being the facts, as a rule differ from the order assigned by the consensus. I have computed the facts for a hundred of the teachers. In computing them I have taken the differences (regardless of signs) between the teachers's ratings and the corresponding ratings by the consensus, and found the sum for each teacher. In this sum *G* does not figure, since the variation was so largely due to varying opinions concerning whether the pupils would have been taught the meaning of the $\sqrt{}$ sign. Differences for *T* and *P* from *either* 7 or 8 (whichever was nearest) were used; differences for *I, X* and *Y* from 18 *or* 19 *or* 20 (whichever was nearest) were used.

The resulting sum of differences from the consensus ranges from 16 to 97, the latter result being rather far on toward the result that would be got from a random shuffling of the problems! The details are shown in Table III.

TABLE III.

INDIVIDUAL DIFFERENCES IN AMOUNT OF DISAGREEMENT WITH THE CONSENSUS IN THE ORDER OF DIFFICULTY OF 24 PROBLEMS.

Quantity. Sum of Differences from the Consensus of the 200 Judges.	Frequency, Percentage of Teachers.
16–23	3
24–31	10
32–39	18
40–47	29
48–55	18
56–63	14
64–71	4
72–79	2
80–87	1
88–95	0
96–103	1

These sums of differences represent (inversely) complexes of (1) ability to judge as the consensus does when it it *right—i. e.,* real ability to judge the "difficulty" of problems, (2) acquaintance with conditions as to the curriculum in algebra that are nearer the average condition, (3) ability to judge as the consensus does when it is *wrong,* (4) time and care given to the ratings, and (5) accidental or "chance" variations due (5*a*) to the individual's condition at the time and (5*b*) to the small number of problems judged.

The last (5*b*) is not a very important cause of the differences found amongst teachers. For the unreliability due to sampling, though fairly large, is small in comparison with the individual differences themselves. For example the two halves of the series (*A* to *K* and *L* to *Y*) gave for the five "best" and five "worst" individuals the following results:

Five Best.			Five Worst.		
Sums of Differences.		Δ	Sums of Differences.		Δ
A to *K*.	*L* to *Y*.		*A* to *K*.	*L* to *Y*.	
8	8	0	39	32	7
11	8	3	31	42	11
8	14	6	35	43	8
11	13	2	44	39	5
17	9	8	39	58	19

And, in general, the mean square error of any one of the entries of Table III. is, so far as the sampling of problems goes, on the average under 5.

It is my opinion that, were all causes of variation save (1) removed, the individual differences would still be so great that the sum of differences from the correct order would still show a range of over three to one.

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MATHEMATICS AS A MEANS TO CULTURE AND DISCIPLINE.

By A. Duncan Yocum.

Perhaps the strongest obstacle to the determination of the disciplinary value of mathematics has been the common assumption that its disciplinary value is not open to question. While the new psychology long since shattered the theory of formal discipline in the sense of the training of generally useful mental faculties, the mass of thinkers still look upon mathematics and the languages as formal disciplines,—as pre-eminently adapted to a mental training that results as a matter of course and is carried over without specific instruction into every field of experience.

Such interesting experiments as those of Professors Thorndike and Bagley on habit transference, with their mathematicians less capable in common judgments than those not mathematically trained, or children trained to neatness in arithmetical papers, whose papers in other subjects remain careless or untidy, have been taken more seriously by educational investigators than by teachers of the formal branches. They prove that a mathematical habit sometimes *does* not carry over. They have not proved that it *can* not, and even if they had, it could not shake the faith of those of us who remain believers in discipline-as-a-matter-of-course, in spite of pupils, who, familiar with the application of general mathematical principles to specific lists of problems, fail to solve miscellaneous examples; or trained to the intelligent demonstration of theorems, are helpless with "originals." Meanwhile utilitarians—sometimes utilitarians in the broad sense—are making the elementary and secondary course of study, selecting the text-books and appointing the teachers, while mathematicians who teach or who write books are tempted to become utilitarian too. All but the specifically practical is being ruled out of arithmetic; algebra is marked for the slaughter; and a pedagogy is becoming dominant which insists

that each subject is useful for its specific value alone. Is it not high time that mathematicians interest themselves in determining more definitely and more certainly the general educational value of the elementary mathematical branches? Whatever this value may prove to be, four classes of mathematical material must be included in the general course of study required of all pupils in the schools:

1. All mathematical material specifically useful to those not specialists, that is sufficiently manysided and recurring in its applications or essential in some unique usefulness, to be made definite and certain for all.

This represents the narrowly utilitarian viewpoint, but sharply emphasizes the distinction ignored by Mr. Spencer between subject matter useful to the race through the specialist, and subject matter directly useful to the majority of individuals who are not specialists. It also makes sharp distinction between material that is specifically useful now and then in a few relationships and material that is frequently useful in many relationships, or uniquely useful in some essential relationship. This last distinction, of unique usefulness, may be illustrated by some link in the demonstration of a highly useful proposition which may rarely recur elsewhere and does not occur in the use of the proposition, but is essential to its demonstration.

The further test of degree of interest, or of sensational or emotional appeal, which in most fields of knowledge must be added to manysidedness and recurrence as a measure of relative value, applies in mathematics rather in determining the applications most useful in furthering ready mastery and retention, than in selecting the mathematical material that is to be certainly memorized and retained. It is also applicable in determining material relatively useful enough to be presented but not relatively useful enough to be made certain through persistent drill, and in determining material in which individuals shall specialize either from the standpoint of occupation or of avocation.

2. The general course of study must include all mathematical material specifically useful to those not specialists which, while not sufficiently manysided and recurring to be made certain for all, is manysided and recurring enough and strong enough in its

sensational or emotional appeal to be presented to all for such individual comprehension and retention as may result.

This distinction, while still narrowly utilitarian, is one that is being persistently ignored by the extremists who are eliminating from the elementary course of study all that is not "useful" in the everyday experience of ordinary people. There is much arithmetic, for example, that is not so manysided and frequently recurring that it should be transformed by drill into habit and system, which is manysided and recurring enough for it to figure as useful information, and strong enough in its sensational or emotional appeal for it to have likelihood of individual and incidental remembrance, retention and association. Carats, grains and pennyweights, par value, even the long-abandoned ad valorem and tare and tret, may be usefully presented to all pupils, even though Troy and apothecary weights are not committed to memory and facility of operation in brokerage or custom-house percentage is left to the specialist.

In determining the relative worth of this class of material, sensational or emotional appeal is not only a factor, but often the determining factor. When the quantity of mathematical material is limited, not by an insufficient supply of what is manysided and recurring, but by the quantity of time available for mathematical study and essential to adequacy and continuity of mastery, some material almost useful enough to be drilled upon, will through its manysidedness ensure the association necessary to incidental retention and the interest favorable to impression, or through its recurrence have an occasional repetition which may hold it in mind. But the sonorous ring of ad valorem, the glitter of the diamond with which carat is associated, the striking visual sequence involved in the determination of the greatest common divisor by repeated division, the motor appeal of cancellation, give a chance of survival more or less useful which would otherwise be lacking. Every pupil has a right to the pleasure—evanescent though definite recollection may be—of seeing some teacher solve an example in cube root or of finding the least common multiple of three or more numbers not readily factored.

Above all, though from the strictly utilitarian standpoint the repeated explanation of a complex mathematical process neces-

sary to rational comprehension is useful only when it is essential to ready use and application of the process, utilitarians must not be permitted to forget the emotional appeal of a cause or a reason once more or less clearly understood. It is this feeling of satisfaction inherent in causal relationships that clearly differentiates them from ordinary associations. It is more satisfactory to know why, than just to know. Pupils may forget why they invert and multiply in dividing fractions or the reason why successive divisors in square root are doubled and multiplied by ten, without lessening their efficiency in practical operations, but the growing impression of the reasonableness of mathematics, vague or partial recollections of general modes of proof and the retention and development of innate interest in causes, from the standpoint of utility furnish stronger incentives to mathematical study than attractive objects or interesting problems; and from that of general discipline and specialization in mathematics as a science, valuable information and ideals more permanent and useful when cumulatively developed through the early school years, than when taken for granted on the beginning of advanced study. Indeed the third class of mathematical material which must be included in the general elementary and secondary course is—

3. Any material, additional to that which is specifically useful to all not specialists, which is essential as a preparation for the more advanced study involved in specialization.

Here it is probably not necessary to distinguish between specialization in mathematics as a preparation for the numerous occupations involving applied mathematics, specialization having as its aim the teaching of mathematics in secondary schools or colleges, and specialization in the pure science with a view to a relative completeness of mastery or of new contributions to some phase of mathematical truth. Each requires a different selection of subject matter in the college or technical school and, to a certain extent, in the preparatory or secondary school course. But each also requires a common preparation for more advanced mathematical work; and one important consideration, too often overlooked by the narrower utilitarian, is the fact that every pupil who has not shown marked inaptitude for mathematics must be regarded as a possible mathematical specialist,

and suffer no lack in his elementary training that will act as a bar to continued study. This condition will be partly met by a course of study where individual aptitudes will be as far as possible determined in the earliest school years, and specialization in mathematics, science or language, given the same encouragement throughout the grades that has long been given, in individual cases at least, to specialization in music and other of the fine arts. But since latent talent is sometimes slow in manifesting itself and cumulative interest is gradual in its development, nothing essential to preparation for advanced mathematical study should be omitted from the general course. From this point of view, as well as from that of utilitarianism and of discipline, there must be a continuity of mathematical study thoughout the entire preparatory period, best ensured through having mathematical study or review every one or two weeks in those years in which mathematics is not given emphasis, in place of a hurried review in the last high school or preparatory school year, which substitutes a partial revival of forgotten learning for adequate retention and gradual growth.

From the standpoint of specialization, several important questions must be answered. In the case of those pupils who have not already specialized in mathematics or who have not shown exceptional inaptitude for it, should not this continuity carry over into the first year of the college course, with a view to furnishing one last opportunity under the stimulus of a new environment and possibly of a different type of instruction, for the discovery of a mathematical ability which has been slow of development? If so, would not the purpose of specialization be better served by the variety afforded through a term in general mathematics consisting of typical portions of various mathematical branches, so limited in quantity as to ensure adequate drill for even those students having the poorest native retentiveness? Could not, and should not, these selections include, both as test and as preparation, the habits essential to mathematical specialization in general? Do the mathematical habits capable of useful application in other than mathematical fields of experience coincide with those most useful for specialization? Whether they do or not, when so much time is spent upon their temporary mastery in the field of mathematics, should not con-

tinuity of mathematical instruction extend itself throughout the college course, in the form of occasional review periods spent in holding in mind the material earlier mastered on account of its specific usefulness to those not specialists or its contribution to general mental training? Finally, should there be a more analytic inquiry into what constitutes mathematical ability, with a view to its earlier detection—especially, if it should reduce itself positively to a strong native retentiveness, whose absence might be balanced by a far more persistent repetition of mathematical habits and systems, or, negatively, to an unemotional or unimaginative temperament which, fatal to specialization in literature, poetry, or art, might with either retentiveness or drill, be less unfavorable to the development of certain kinds of mathematical efficiency? The answer to all of these questions lies in analysis of mathematical discipline into definite complements of knowledge and mental activity.

The fourth class of mathematical material which must be included in the general elementary course of study is

4. All mathematical material additional to that which is specifically useful to those not specialists or to that essential as preparation for specialization, which is essential to general mental training.

Here the discrimination is between material useful both specifically to all individuals and as an essential phase of general mental training, and material specifically useless as mathematics except through the specialist, but essential to mental training in general.

The question involved is not whether mathematics affords mental discipline that can usefully carry over into non-mathematical phases of experience, but whether all that can be usefully carried over, cannot result from the thorough teaching of those portions of mathematical material that are specifically useful to the general student, and whether the conditions favorable to its carrying over are found in mathematical teaching. And if not, since it can be applied in other fields, whether it cannot be more economically developed in some other field, or developed with greater certainty of the conditions favorable to its general application.

Is it possible to teach smaller portions of arithmetic, algebra

and geometry than are commonly taught? Can adequate mathematical and general discipline result from the teaching of a few selected portions? Can it result from the partial study of some one mathematical branch? Can the partial study of one mathematical branch be as adequate preparation and incentive to mathematical specialization, as a more partial study of two or three? Will such a partial study, if practicable from the standpoint of interdependent operations and demonstrations, be less disciplinary or more disciplinary? What proportion of the subject matter of mathematical branches as wholes is necessary to cumulative system as a factor in discipline? And for what proportion of mathematical material is certainty and continuity essential to usefulness? Will more partial subject matter lend itself more advantageously to both continuity and system as dependent upon occasional review? Are the conditions favorable to general application within the mathematical field itself inherent in elementary mathematical teaching? If not, does the thoroughness of mathematics as a specific discipline necessarily ensure its usefulness as a general discipline? All of these questions are basal for the determination of the place of mathematics in the general course of study. None of them can be answered until mental training is definitely analyzed into its several fundamental phases, and the contribution of mathematics to each has been specifically determined.

There is a difference between a psychological analysis of mental activity or a logical analysis of culture, and a pedagogical analysis of mental training as including both discipline and culture. Formal self-activity is a broader thing than formal discipline—in the sense of developing a mental faculty. Educationally, it is necessary to separate out from it those phases of self-activity which further the development of all others. They must readily reveal themselves when effort is made to determine what is left behind after any kind of subject matter has been presented.

Parallelling the relationships in which the subject matter is retained are at least five phases of formal self-activity which include all that can be called mental training, discipline or culture. First, most ideas or experience, once presented or even mastered, are forgotten altogether,—their sole permanency be-

ing in feelings and impressions that are educational only when they cumulatively center about some useful idea or action. Opinions, points of view, prejudices, interests, ideals, what is collectively called public opinion—are all the product of this *cumulative impression* largely based upon forgotten experience.

Second, most experiences which are not wholly forgotten except as impressions are merely held in mind in a single or very partial relationship, varying with individuals, incidental, and often false or non-essential. The chief means to this *mere remembrance* are words standing for very partial concepts, but constituting centers for association or apperception and so conditioning the growth of mental content. It is, consequently, furthered by all vocabulary expansion which retains ideas and words with a minimum of association and comprehension.

Third, a certain proportion of ideas and words that are merely remembered are gradually associated in a manysided way with other ideas, the association and manysidedness varying with individuals according to their past experience, their more permanent or more immediate interests, and changing moods and experience. The chief characteristic of this *varying apperception* is its variability, individuality and growing manysidedness. Its formal value lies on the one hand in making partial concepts more adequate and useful concepts stronger, and on the other, in bringing about such mental interconnection that any idea which does not linger in the stage of mere remembrance may be associated with any one among a multitude of other manysided ideas.

So far as culture has not been identified with discipline, it very largely consists of this varying manysidedness, a broad vocabulary and cumulative impression, related to the spiritual inheritance of the race, including æsthetic appreciation, and common to all educated individuals who enter into social intercourse with each other.

Fourth, a relatively small proportion of the material of experience and instruction, limited physiologically and psychologically, can be made definite and certain as *specific discipline* through habit and system. Its main condition is continuity of subject matter in unvarying relationship. Therefore, from the limited time subjectively possible not only for memorizing but

for review, such material can include only the relationships most useful to specific preparation for life or to general training in the form of cumulative impression, mere remembrance, varying apperception, and the conditions, yet to be discussed, favorable to general discipline or application.

Fifth, the ideas and activities so highly useful that they are made permanently habitual, should be given as general an application as is possible and useful. The possibility of this *general discipline* depends upon a stimulus and consequence general enough to be found in other situations or fields of experience than the one in which their relationship became habitual. Hence, the more specific the system which constitutes a banch of learning, the more numerous its specific details and habits which are definitely useful only in their own field, the less likely it is to result in general discipline. The more thorough the specific discipline and the more complex its system, the more wasteful it is as a means to general discipline, if a multitude of comparatively useless relationships must be made habitual in order to certainly develop a few for which a general and useful application is possible.

More than this, the thoroughness and continuity through which a specific discipline may ensure the habit that should be generally applied are only the first of many conditions favorable, or even essential, to general discipline. The varying manysidedness and complete mental interconnection, impossible to an abstract, specific discipline, are essential to general application. The more useful and more general the habit to be applied, the more essential a potentially variable apperception is. Especially in the case of moral habits, the general stimulus should be emotionalized through cumulative impression. Far from following as a matter of course from the mastery of some of those branches through which varying apperception and cumulative impression are ensured, other conditions favorable to general application constitute in themselves a specific discipline, a definite system altogether outside the discipline and system of an academic subject. Typical applications in other fields should be certainly associated with the general stimulus. The habit of looking for varied applications should be formed. Specific knowledge peculiar to each possible field of application must be gained, adequate

to the identification of the stimulus and its application. Above all, the habit of analysis and synthesis must be formed in the most useful fields, or in any field on the recognition of some part of a complex stimulus or a well-memorized sequence of possible stimuli.

More than this, without the conditions just enumerated, general application of an idea or habit within the very academic field in which it is formed, is uncertain and even improbable.

If the cultural and disciplinary value of mathematics is to be determined, it must be through a complete analysis of mathematical subject matter in relation to the furtherance of each one of these five phases of formal self-activity. In a limited discussion such as this, analysis can be carried only far enough to indicate the line that it must take, or at best, to suggest far-reaching conclusions that only complete analysis can verify or disprove.

From the start, distinction must be made between cumulative impression, mere remembrance, varying apperception, and specific or general discipline that further the study of mathematics, and mathematics as furthering them. Ideals and feelings, vocabulary expansion, information getting, specific knowledge and habits, and the conditions favorable to general discipline, all are more or less essential to mathematics. The question to be determined is the extent to which mathematical study is essential to them. If it is not obvious that no material not mathematically useful should be included in mathematics on the ground that it is useful from some other point of view, it is at least obvious that no material must be included whose mastery impedes mathematical teaching itself. On the other hand, no mathematical material, no matter how useful to the specialist, must be included in the general course, that is not specifically useful to those who are not specialists, unless the specifically useful fails to include material essential as the elementary preparation for future specialization, or to some one of the five phases of formal self-activity, which cannot be more effectively, economically or usefully gained elsewhere.

In the first place, mathematics is a potent means to the development of cumulative impression which may or may not carry over to other fields of experience. Two of the most prominent

mathematical ideals are those of exactness and of persistence in the face of complex difficulty. Both are useful in other fields, but differ vastly in extent of usefulness and probability. The ideal of exactness can readily be carried too far, if it is not checked by a strong sense of the usefulness of partial concepts. It must not be confused with the ideal of truthfulness, because exact statements are often untruthful abstractions, lacking the perspective, the concreteness and the emotional understanding which must modify the matter of fact. On the other hand, the ideal and the habit of persistence in the face of complex difficulty, highly useful to specialization, a factor in will training, and less dependent than most ideals upon the specific thing that develops it, is little likely to be formed in an elementary arithmetical course that has eliminated everything complex from fractions to evolution and partial payments. On the strength of the fact that it is essential to specialization and the emotional re-enforcement for a generally useful habit, a complex example should occasionally be solved or even a complex process temporarily mastered. To be sure, the same interest in the complex can be developed elsewhere, but the great mass of children now in the elementary school are not likely to develop it unless they test their comprehension and persistency by now and then working their way through a complex or compound fraction, or solving an example in cube root. Even though they forget how to do both the next day, the impression will remain if they are repeatedly made conscious of the incentive to effort which should come through the challenge of a complex difficulty. The same thing is true of the so-called puzzle problem, now totally ruled out of court on the ground that it is not specifically useful. Here again specialization demands an ideal which utility will not afford. Knowledge or achievement for the sake of knowledge or achievement, assigned to all but not required of all, specifically remembered by none and wholly omitted from examinations or tests, will at least serve as one means of selecting the pupils most likely to become specialists, and of cumulatively developing the love of pure science. Reference has already been made to a similar instance of a permanent interest not essential to specific utility,—the pleasure of knowing the reason for things as the cumulative means to a permanently rational attitude of mind.

Utility demands that the pupils shall be drilled into more or less permanent comprehension of only those processes and judgments whose use or application will be furthered by familiarity with the reason for each step. The possibility of future specialization, and general discipline demand that pupils shall have the reason for every new operation explained to them, not in order that they shall remember it, but that they may become increasingly conscious of the fact that there is a reason for all things.

Cumulative impression up to this point has well emphasized in the fields of specialization and discipline the distinction already drawn between that which all pupils must certainly master and retain, and that which while presented to all, is left wholly to individual and varying mastery. Even this varying mastery, essential to mere remembrance and varying apperception, is unnecessary to cumulative impression, where the repetition of a feeling gradually emotionalizes an idea or activity.

Another condition unnecessary to cumulative impression but favorable to mere remembrance and varying apperception is the manysidedness or recurrence of the terms and experience mastered in specific problem work. Here the object of impression is merely immediate interest in successive applications until, through accumulation, permanent interest is acquired in mathematics in general or the particular phase of it applied. It is non-essential whether or not the subject matter of the problems is good apperception material, likely to be remembered or to aid in remembering and associating. The essential factor is an immediate incentive strong enough to re-enforce a cumulative interest. In fact, the more this interest is carried over to the material of application, the less centered it is upon the mathematical process itself. From the standpoint of cumulative impression, therefore, the arithmetic which substitutes information-giving examples for applications in the immediate experience of the children, overshoots its own mark. The main incentives which, in addition to a love of exactness or perseverance in a complex task and fondness for unravelling a puzzle, are not only immediate, but concentrate interest upon the work itself, are pleasure in mechanical operation, the satisfaction that comes with ability to do, and the realization that each application is of immediate use. Where, as suggested in the sane and

definite report of the National Committee of Fifteen on Geometry Syllabus, problems deal with applications by specialists in remote and highly technical though interesting fields, the interest in the application may be strong enough to carry back to the principle if, as the Committee itself cautions, they do not "contain technical terms and mechanical terms and mechanical features unknown to the average pupil and not easily understood without more explanation and consequent distraction from the geometry itself than is warranted in the ordinary course." This caution applies with equal force to Dr. Frank McMurry's suggestion that the function of arithmetical problem work in the higher school grades is to "identify the pupil in knowledge and interest with his business environment; or perhaps better, with his environment on the quantitative side." Sharp distinction must be made between mathematics as a means to interest in other things, and the use of other things as a means to interest in mathematics. From the latter standpoint the use of interesting applications drawn from specialized arts and industries is made exceptional, first, by the fact that most of the essential and permanent interests can be developed through material specifically useful to those not specialists and should be associated with it; and second, where material is included for the sake of the ideals essential to the mathematical specialist, the incentive of immediate usefulness or interesting application must be displaced by a love of pure science. This limits the use of any applications useful only through the specialist but interesting to the general student, to the few geometrical propositions which, though not directly useful, condition the mastery of those that are, and confines selection to applications which combine the highest interest with the least amount of non-mathematical technique. A limited number of arithmetical impressions then may cumulatively come to have general usefulness as ideals, opinions, interests and incentives. But it must not be inferred that every generally useful idea or practice incidentally results in an ideal. An instructor may demand and secure exactness or neatness without pupils developing a *love* of exactness or neatness. Unless some strongly impressive incidents or experiences are certainly associated with it as an *emotional center* about which impressions ensured through in-

struction gradually accumulate, the vague feeling may never become a conscious motive. Moreover, while an ideal is more likely to carry over into new fields of experience than a non-emotionalized idea or activity, to ensure the greatest likelihood of its being carried over, the remaining conditions favorable to general discipline and application must be ensured.

To sum up, the few mathematical ideals that may be generally applied are for the most part useful in all the mathematical branches and may be developed by the use of a limited amount of material in any one of them. This material might exclusively consist of what is specifically useful to direct preparation for life, and mainly of what is not made permanent through persistent drill, were it not for a still more limited number of ideals or incentives, fundamental for specialization, that are more likely to be effective if they are the cumulative products of a development that has been continuous throughout a long period of time. This justifies the addition to specifically useful material of a small amount of material useful only in the development of incentives essential to specialization, as the result of its occasional and conscious use for this end alone, rather than of its thorough mastery

The discussion of the extent to which mere remembrance or vocabulary expansion furthers mathematics, and mathematics furthers it, need not be highly analytic. Mathematics is largely an abstract science. Its vocabulary is not only limited, but largely limited to words that have only mathematical use. For the most part, it furnishes words to be remembered, rather than words so manysided and recurring that the student can remember others by them. The chief opportunity for general vocabulary expansion through mathematical study, lies in miscellaneous and manysided application in fields that abound in useful terminology. Here distinction must be made between the abstract and all-inclusive mathematical terms expressing general numerical relationships, and more specific terms or applications that *can* be associated, or at least *will* be associated, with a more limited group of things. There is absolutely no limit to the application of general numerical terms and processes. All things can be grouped together by common number or measure. But a general mathematical relationship is too abstract to hold in

mind words thus associated. The way in which mathematical problems expand vocabulary is in associating together things that could just as readily have been associated without being mathematically related, and which will be retained in association with each other no more permanently on account of their mathematical relationship.

It is none the less true that mathematical problem work *furnishes an occasion* for such association, and that if many-sided and recurring terms are in consequence remembered, vocabulary formation is furthered. Against this use of general mathematical terms and processes, however, lies the fact that outside of the amount of application necessary to their mastery, this vocabulary expansion is non-mathematical, and the further fact that wherever adequate application is possible with familiar words, the mastery of new terms not mathematically necessary is an obstacle to mathematical progress. Moreover, while a limited familiarity with the vocabulary of a particular field is essential to a mathematical application within it, instruction can concern itself with the vocabulary of only a few typical fields in which application is certain to be made. The mastery of vocabulary in general or information in general, so helpful to generally useful habits, whose application is dependent upon the identification of their stimulus in some unexpected situation, is not essential to mathematics. Where mathematical application is necessary, it is compelled by circumstance in some one or another of a thousand fields, whose vocabulary is already familiar, or must be mastered for the same reason that requires the application.

From the standpoint of mathematical progress, the words used in ordinary problem work should be, so far as possible, already familiar to the pupils, unless interest or incentive can be gained only through the bringing in of new terms readily understood, or application in a typical field which requires a special vocabulary. In the one case, immediacy of interest is the determining factor in their selection; in the other, specific usefulness. In neither is there any assurance of the manysidedness essential to the words that tend to hold others in mind.

The mathematical terms and processes which are more limited in their applications and associations are better adapted to hold-

ing new words in mind. These are the cases where, as mathematics finds its application in some specifically quantitative phase of life, words are used that express actual mathematical relationship. Some, like *triangulation* in surveying, are little likely to enter the ordinary vocabulary except through mathematics. Others, like *capacity* and *volume*, will be acquired through other fields of knowledge.

It clearly follows that no branch of mathematics will be studied as a means to vocabulary development, and that vocabulary development or mathematical study, if it is not to interfere with mathematical progress, must be restricted to mathematical needs.

The probable truth of this conclusion becomes more obvious when mathematics is considered as a means to information getting, or the development of the basis for varying apperception. All that has already been said in connection with mere remembrance or vocabulary formation, holds true here with even greater force. In place of complete mental interconnection, which only varying apperception ensures, so essential to all application dependent upon the recognition of an accustomed stimulus in some unfamiliar form, the applications of mathematics in ordinary life usually arise from external necessity. Recognition of an occasional stimulus does not suggest mathematical operation, but the necessity of mathematical operation compels search for a familiar stimulus. General knowledge and information are, therefore, no necessary part of the mathematician's equipment, but rather knowledge specifically necessary in some typical field of applied mathematics, or knowledge that must be newly acquired in order that some obvious application can be made.

On the other hand, in spite of an ingenious and stimulating Herbartianism that is striving to make every branch of knowledge a means to apperception, mathematics is the medium most unfit for the acquisition of information and the completer interconnection of mental content. If Professor McMurry had stopped with his assertion that the principal purpose of such arithmetic as is taught beyond what is essential to efficiency in the fundamental operations, "should be, not to teach processes, but to identify the pupil, in knowledge and interest, with his business environment," he would have emphasized an absolutely

non-mathematical phase of work. When he goes on to say—"or perhaps better, with his environment on the quantitative side," he emphasizes mathematics, but emphasizes it in a relationship in which concrete comparison is of far greater value than any amount of exact figuring. It is only partly true that "what a railroad system really is can be understood only when one comes to understand how many men it employs, what income they receive, how long they have to work, how many shipments are billed per year, what quantities of goods are transported, how much capital is invested, what profit the stock brings, etc."; or that "one must know the actual quantities involved in order to appreciate what a warship means, what a farmer does, what a mine is." A vivid piece of prose using such concrete comparisons as the number of days it would take the whole army of railroad employees to pass a certain point, or the number of city blocks a warship is long or number of acres or squares comprised in a warship's decks, would leave a more useful and impressive concept behind than any amount of multiplying or dividing or finding per cents. The college boy soon forgets the formulæ he used in his astronomy, but is likely to remember that the moon revolving about the earth placed in the center of the sun would still be far within the solar surface.

Over and beyond all this, the quantitative phase of some useful idea is always too definite and specific to associate itself with other things or have other things associate themselves with it, except through comparison that need not be exact. It is something to be remembered, not something to remember by and think with. You may associate with angle the notion of right angle or equality and within the mathematical field it may call to mind a perpendicular line, a right triangle, equality of triangles, and other groups of mathematical ideas already familiar and necessary. But associate a new idea in some general geographical or historical location or sequence, such as China, the Franco-German frontier, the mountains, the age of Elizabeth, the French Revolution or Chivalry, and a host of ideas come crowding into mind, all associated with it by contiguity in time or space, and many associated with it in necessary relationships which without its general location might remain undiscovered. A group of mathematical relationships cannot reach out into

life in general for new information and experience. But once certainly associate with such a general term as natural product, a suggestive sequence of ideas—source, location, mode of production, the uses to which it is put, and each new name to which it applies demands new inquiries and stimulates new interests.

It is manysided centers of association such as these that multiply knowledge and broaden interests, not mathematical abstractions or terms incidentally involved in mathematical applications or brought into mathematical problems to make them interesting. No one who carries his analysis far enough into the mode by which vocabulary is expanded, information increased and mental interconnection multiplied, would urge as a phase of it the teaching of any mathematics not in itself specifically useful, or for the learner who is not a specialist, any mathematics more advanced than arithmetic or any phase of arithmetic beyond the fundamental operations. Even in the preparation of possible mathematical specialists, there are probably no terms, processes or applications whose early mastery would later aid in the acquisition of mathematical vocabulary or information, except the use in elementary work of an occasional technical term, that might otherwise be popularized or juvenilized.

It is only when mathematical material is regarded from the standpoint of specific discipline—its certain formation of habits, and their interconnection in definite and complex system, that its peculiar and traditional disciplinary value stands revealed. No subject matter demands greater certainty and greater system—with each detail in more definite relationship, or system more inclusive of definitely related details. Other subjects *can* be taught as systematically as mathematics, but mathematics *must* be systematically taught. Any mathematical habit that can be applied in other fields *may* be certainly developed in other fields, but in mathematics it *must* be developed. Just as a study of Latin and Greek which includes mastery of their literatures, ensures a manysidedness and culture which a specialization the college course at present permits can fail to gain, so the mastery of mathematics ensures a specific discipline which as yet rarely results from the teaching of any other subject, or from preparation for any definite phase of life.

To be sure, this superiority is but temporary. Just as certainly

as the sun rises and sets, will a more complex, more certain and more continuing system result from the determination of relative usefulness of definite details within the various branches, both to the formal phases of self-activity now being discussed, and to direct and specific preparation for morality, health, industry, social service, citizenship and social and individual leisure.

It is this readjustment already under way that makes it necessary to take careful account of mathematical stock. The most suggestive fact to hold in mind is that while any system can be made certain, the part of school life that can be spent in building up certain and specific habit and system is physiologically and psychologically limited. Now mathematics as system is essential in the general elementary school course only in so far as it is specifically useful as mathematics to the general student, or necessary to the formation of generally useful ideas, ideals or habits. Analysis of cumulative impression has shown that some material and instruction not specifically useful to the general student must be included for the sake of ideals and interests necessary in the preparation of specialists. But with the exception of a few emotional centers, it need not be certainly memorized, and so does not belong to specific discipline. The analysis of mere remembrance and varying apperception has shown that while mathematics makes incidental contribution to vocabulary expansion and information getting, it contains no memory or apperception centers useful enough when brought to bear upon non-mathematical material to be certainly mastered. The analysis yet to be made of the conditions favorable to general discipline or application, can do no more than add certain non-mathematical system, as helpful to the carrying over of mathematical ideas or habits to non-mathematical fields in which they are useful.

The chief remaining question then, as regards specific discipline, is what part of the system peculiar to each elementary branch of mathematics is specifically useful to the majority of individuals who are not specialists? Of course in the limited time available, this analysis cannot apply the test of relative manysidedness and recurrence of specific usefulness, or unique specific usefulness, to the details of arithmetic, algebra and

geometry. In the case of arithmetic, elimination has probably gone too far—at least in its failure to retain material too little useful to memorize, but useful enough to present for individual comprehension and retention and in some cases essential to ideals and incentives preparatory to specialization. The failure of mathematicians to answer the utilitarian's challenge that algebra is almost wholly lacking in specifically useful material, shifts the question to its value as a general discipline and the extent to which its mastery is essential in the discovery of specialists, and elementary preparation for specialization. The fact that the sources for practical geometrical problems recommended by the Committee of Fifteen, are mainly mechanics, shop-problems, steel manufacture, geometrical tracery, Gothic architecture, pattern design, geometrical ornamentation, Mosaic pavements, and decorated windows, leads to a similar conclusion.

Assuming that very small portions of algebra and geometry are specifically useful to those not specialists, but two questions remain: how much mathematics, not thus useful, is necessary for the discovery and preliminary training of mathematical specialists; and how much is essential to the formation of generally useful mathematical habits. The answer to the latter will at least partly complete the answer to the first already begun for ideals and incentives.

In the first place, there appear to be no habits in algebra and geometry, generally useful in non-mathematical ways, that are not to be found in arithmetic as well. The habit of exact thinking, of successive or cumulative judgments or conclusions, of analysis with a view to the identification of a familiar term, of instantly calling to mind the combination or series of combinations the identified term should suggest, of analysis with a view to discovering the remaining terms in the combination which results in a judgment, of perseverance in the face of difficulty or complexity, of looking for varied applications of a principle, of seeking the reason for each process or judgment,—all these are not only common to the three subjects, but so fundamental for each as to constitute the chief equipment for specialization. All can result from the study of arithmetic alone, and, with the exception of the seeking out of reasons and perseverance in the face of complexity,—of such parts of arithmetic as are specific-

ally useful to all not specialists. Even from the standpoint of the certainty favorable to general discipline, the study of any one of the elementary mathematical branches as a whole, or of more than one, is totally unnecessary. Concentration upon related parts, rather than completeness of system, is the favorable condition to fixed habit.

All that is essential to the discovery and preliminary training of specialists, in addition to those ideals and habits generally useful in mathematics, including the ideal and the habit of perseverance in the face of complexity, and of seeking the reason for all processes and judgments, is such experience with algebraic and geometrical material and operation as will reveal peculiar adaptation or lead to special interest. Would not simple equations and the fundamental algebraic operations and a book of plane geometry or selected portions from more than one book adequately serve this purpose? Furthermore, should not as large a proportion as possible of this algebraic and geometrical material be left to individual retention, rather than be drilled upon until certainly mastered? To illustrate, definite recollection of the solutions given in the text-book for particular theorems is less essential than the retention of the theorems and mastery and ready use of the more common combinations, sequences and judgments solutions involve, aided by the generally useful mathematical habits already gained. Even the material thoroughly drilled upon and the habits certainly developed through phases of mathematics that are not applied in the ordinary experience of ordinary people, lack the continuity necessary to permanent certainty, except in the case of the specialist.

Finally, the extent to which mathematics furthers general discipline or application and is furthered by it, depends upon the extent to which its mastery includes the conditions favorable to the carrying over of these generally useful habits into other fields of experience, to the carrying of them over from other fields of experience to mathematics, and to the carrying over of all mathematical habits and judgments to the specific mathematical problems or operations in which they are applied.

The few of these favorable conditions which are general are favored less by mathematics than by many other subjects. The cumulative impression which idealizes them and gives motive for

application is less emotional in the case of mathematical subject matter, and tends to purely mathematical application. The manysided material, which through varying apperception provides the completeness of mental interconnection essential to the most general and remote application, is wholly lacking. The habits of making new applications and of analysis following the identification of any part of a complex stimulus with a view to the identification of the whole, are confined to mathematical material. The favorable conditions which are specifically related to typical fields of application, will of course be specifically mathematical, or specifically outside the mathematical field. This is the case with certain association with the habit, of a few typical applications, and the knowledge necessary to application in the most useful fields.

If mathematics has as yet failed to ensure the presence of the conditions favorable to general application within the mathematical field itself, there is small reason to include in its system, conditions favorable to more general application that naturally belong in the complex systems of essential ideas and habits through which morality, health, industry and good citizenship will before long be taught. It is probably more economical and rational to ensure these habits as early as possible in general education, with the conditions favorable to their being carried over to mathematics, than to use them as theoretical justification for the mastery of elaborate phases of mathematics, not otherwise useful, in which the conditions favorable to more general application are lacking. At least it is safe to assert that no part of mathematical material or system should be included in the general elementary curriculum on the ground of discipline alone.

The correction, then, of present weakness in the teaching of mathematics lies less in the devotion of more time to the subject, or concentration upon some branch of mathematics as a whole, than in the more effective teaching of those portions of arithmetic which are directly useful, plus the little arithmetic not thus included and the parts of algebra and geometry, which are essential as preparation for future specialization. This should involve a more thorough mastery of mathematical habits generally useful, and the conditions favorable to at least their general mathematical application.

With the study of algebra and geometry as wholes left to those students who specialize early in mathematics, ample time will be left throughout the school and college course for the occasional review necessary to continuity for what is essential to all and favorable to specialization by the few whose mathematical abilities and interests are undiscovered or undeveloped until the beginning of college life.

Incidentally, specialization can be more thorough in advanced classes which contain no students who take mathematics through compulsion as Fluellan ate the leeks, while no pupils uninterested or non-proficient in mathematical study, will find it a bar to further advancement in the common culture and discipline essential to true democracy.

In a book with which a few of you are familiar, I have much more fully carried out certain phases of this analysis. Whether you agree with my conclusions or not, I believe you will agree with my mode of approach to these fundamental questions, and with the fundamental viewpoint which characterizes it—the belief that the vital questions it seeks to answer must no longer be expressed in the vague generalizations which encourage controversy, but in definite and specific terms and conceptions, whose relationships and consequences are the product of the same patient and unemotional investigation and analysis through which the true mathematician studies the science which he loves, and in the case of many individuals are less likely to be analyzed out or understood in the absence of some of the ideals and habits which mathematics develops.

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WHAT MATHEMATICAL KNOWLEDGE AND ABILITY MAY REASONABLY BE EXPECTED OF THE STUDENT ENTERING COLLEGE?

By James N. Hart.

If we contemplate the numerous committee reports of this and other societies upon the teaching of arithmetic, algebra, and geometry, together with the excellent syllabi published by them, it would seem that our question has been so fully and definitely answered, that further discussion is uncalled for, if not almost impertinent. If, on the other hand, we turn to the series of papers presented before the Association of Teachers of Mathematics for the Middle States and Maryland last year and printed in the Mathematics Teacher for March and June of the present year upon the question, "What Mathematical Subjects Should Be Included in the Curriculum of the Secondary School," it is very evident that the doctors still disagree. At one extreme we find a speaker, a college graduate, but not, according to his own confession, a mathematician, arguing that so few students are "mathematically minded" that it would be more profitable to limit high school mathematics to arithmetic and allow the few who reach college to elect algebra and geometry—if they have any remaining curiosity regarding mathematics. At the other extreme, a high school principal presents an "outline of mathematical work that should be required of every student in a general high school course," including not only the traditional work in algebra and plane geometry, but solid geometry, trigonometry and applications of algebra to mechanics, science, economics, statistics, shop mathematics and the slide rule. And again, when we consider the propositions of the committee on articulation of the N. E. A. and those of the commissioner of education of the state of Massachusetts we realize that the question is not finally settled, and that we mathematicians will not be allowed to settle it by ourselves.

Nearly, if not quite, all the colleges of New England have

adopted the definition of the mathematics requirements formulated, I believe, by the College Entrance Examination Board. We all assume that admission to the study of high school mathematics implies the completion of arithmetic and, naturally, entrance examinations in this subject are no longer set. I sometimes wish they were. These uniform mathematics requirements have been in force so long and have been so freely discussed that one might think that there could no longer be any doubt as to what the colleges expect of the entering student, mathematically. And yet we all know that what they get is still an exceedingly variable quantity,—we might almost say a function of an indefinite number of variables.

In the grades the pupil studies arithmetic for from 6 to 8 years; in the high school he has from 1½ to 2½ years of algebra, 1 to 1½ of geometry and, if he is looking toward a technical course, he adds solid geometry, and trigonometry. Thus he has nearly continuous instruction in mathematics from the beginning of school to the beginning of college. Would it not seem that he should be at least as well equipped in this subject as he is in Latin, which he studies four years, or French, which he has for two or three?

In the *Educational Review*, December, 1911, appears an address given before the American Institute of Instruction by the President of the New England College Entrance Certificate Board. In it are statistics showing the great decrease, under the certificate system of admission, in the percentage of students having an unsatisfactory record during their first semester in college. Figures are given for various subjects for the year 1903–4, and for four years later as follows:

	1903–4.	1907–8.
English	20.2%	7.8%
Latin	14 %	3.7%
Greek	11.8%	2.9%
French	14 %	6.6%
German	13.4%	7.9%
Mathematics	24.6%	13.6%

In all subjects except mathematics and German, the percentage of unsatisfactory records in '07–'08 was much less than half that of '03–'04. Mathematics shows a great improve-

ment, but less than any of the languages, and its percentage of failures is more than double the average for the five language subjects. The last annual report of the Board shows a decidedly improved percentage of failures in mathematics, but still more than double that in the languages, namely, 10.8 per cent. of failures among those admitted by certificate who continue mathematics in college as against 4 per cent. of failures in language. According to the same authority the percentage of failures in mathematics of those admitted by examination was 19 as compared with 9.1 per cent. of those who offered the languages and continued the same language. We still see the percentage of failures in mathematics more than double that in language. Now, these figures are from the product of picked schools, most of which have stood the test of repeated reviews by the board, and the students that have entered college on their certificate have also been carefully selected, only those whose ranks are of A or B grade being, in general, granted a certificate. Except in case of ill health, and such cases would naturally not be reported to count against a school, why should there be any failures to make good, and, in particular, why should mathematics have a larger mortality than language. Almost one ninth of the certified pupils and nearly one fifth of those examined and taking mathematics in college fail to pass. At my own institution the figures are similar, our percentage of failures being somewhat larger. Last year our freshman mathematics for the fall semester was a combined course in algebra and trigonometry and practically one man in four failed to make passing rank. In the college of arts and sciences the record was better, only one in eight falling by the way. This marked difference in the records of our students in our different colleges is probably due to two causes:

1. The crowded curricula in engineering and chemistry, calling for about 28 hours per week of attendance upon college exercises, 13 of which are recitation courses requiring, theoretically, at least, two hours of preparation for one of recitation, making a total working week for the freshman of 54 hours.

2. The fact that many students begin a course in engineering who are not adapted to the work and who, in a college where mathematics is elective, would leave it alone. A considerable

number of these learn their own limitations in this direction and seek a path of less resistance at the end of the first semester.

Who is to blame for this regrettable mortality in freshman mathematics,—the freshman himself, his high school teacher of mathematics, his college teacher, or the system of education? No doubt all of them are responsible, perhaps the high school teacher least of all. The change from high school life with the restraining influence of parents and teachers who have long know them, is a perilous one for many young men, and their inability to use this new freedom discreetly accounts for some of the failures.

Some failures are due, no doubt, to the lack of sympathy between the freshman and his mathematics teacher. The college administration too often feels obliged to put some of these classes in the hands of a graduate student or fellow, or, at best, a teacher of but little experience. The freshman needs, and should have, the benefit of the best teaching talent upon the faculty. At the University of Maine it is our practice to have every member of the mathematics department whose schedule permits it, teach some freshman work. Five of the seven men at the present time giving instruction in freshman mathematics had considerable teaching experience in preparatory schools before taking up college work.

In order to promote closer acquaintance with the students each member of the department holds one hour or more per week open for those who wish to consult him about their work. The students are not only invited, but often required, to attend these conferences. Some of the instructors spend much more than the allotted time with the men. At these conferences an attempt is made to find out what hinders the student's progress, —whether he lacks preparation, is studying too little, or not understandingly, etc. Those needing more help are, of course, advised to tutor.

The constant temptation of the college teacher is to overlook the fact that the freshman's mind is but little more mature and can work but little faster than when he was in the high school. Algebra, trigonometry, elementary analytical geometry, solid geometry, all are so simple to the teacher, why should the student have trouble with them? Perhaps he would not if he really had

the preparation that our stated requirements call for, or assume that he possesses. And this brings me to my subject,—*What Mathematical Knowledge and Ability May Reasonably be Expected of the Student Entering College?*

He has graduated from a school approved by the certificate board or by the department of education of his state, has presented either by certificate or by examination, the full number of units required for admission, including the usual requirements in mathematics; is he ready to enter upon the course in college mathematics with no backward look upon the path he has already traveled?

Let us state some of the things he must know and some of the things he must be able to do if he is to go on successfully. No attempt is here made to follow a logical order or to make the statement complete. First of all, he must have ready for instant use a knowledge of the fundamental facts of arithmetic and must be able to perform its processes with fair rapidity, and be sure that his results are correct. Many students fairly good in written work in algebra are painfully deficient in the solution of easy oral exercises. Personally, I wish that there might be in the high school curriculum a place for the old-fashioned mental arithmetic. I presume that the pages of oral exercises given in our modern text-books of arithmetic and algebra are supposed to take its place, but I fear that many teachers skip these pages. Not long ago a young lady having the reputation of being "the best teacher in town," was conducting a class in arithmetic in the grammar school. The text, an excellent one, was well supplied with oral problems. When asked if she found these exercises helpful she replied: "We do not get time for them." A few minutes later a boy was sent to the board with a problem,—"the area of a circle is 314.16 sq. ft. Find its radius." The pupil divided by .7854, then extracted the square root, then divided by 2. Another boy asked "Why not divide by 3.1416 and extract the root?" The teacher replied: "We had better always follow one rule." Would not she have done better to spend more time on the oral exercises?

Before leaving the study of arithmetic, pupils should be taught the abbreviated processes of multiplication and division and should be required to practice cancellation and a few other

"short cuts." According to my observation, comparatively few freshmen will divide by four and moving the decimal point instead of multiplying by two hundred fifty, or multiply by 100 and subtract twice the multiplicand instead of multiplying by 98, or choose the easier multiplier when required to find the product of two numbers of, say five figures each, one containing ciphers or repeated digits. Pupills should be required to make rough checks of the reasonableness of their results, particularly in problems involving the placing of the decimal point. If these things have not been done in the grammar school, they should be attended to in the numerical work of geometry and algebra.

In algebra, the aim should be not so much to cover all the ground mentioned in the college catalog requirements, as to master thoroughly the principal subjects, to teach the pupil to reason out all processes and to frequently test results.

Many freshmen have no precise knowledge of the meaning of algebraic terms, but will confuse index and exponent, term and member, and allow the one word simplify to "cover a multitude of algebraic sins." Some of the most frequent of these sins are: cancellation of a term in the numerator and a term in the denominator of a complex fraction, multiplying one member of a fractional equation by the common denominator of its fractions leaving the other member unchanged, writing $a + b$ for the square root of $a^2 + b^2$.

The pupil who can factor the forms commonly met, change simple surds, whether written with radical signs or fractional exponents, from a given form to another desired form, solve simultaneous equations in one, two or three unknown quantities, form and solve the equations for problems of moderate difficulty, including such as involve radicals and quadratic equations, expand simple expressions by the binomial formula and do these few things rapidly, and with confidence in his results, has the ability to undertake freshman algebra with good prospects of success. The pupil should have learned to read and understand a text-book without the help of a teacher to translate it for him.

In the second course in algebra, or the senior review, it appears to me desirable to teach the use of logarithms in connection with the study of exponents. The pupil will be pleased to learn such a convenient device for finding products, quotients,

powers and roots. I would have this done, not so much for those who are going to college as for the larger number who are not.

In geometry, also, the emphasis should be placed upon quality rather than quantity of preparation. Many of the theorems printed in the average text-book may profitably be omitted. As to the matter of "originals," I should not, perhaps, venture an opinion, as my own preparatory work in geometry was done many years ago with a text-book giving but few of them. But having long felt that the solution of originals was being emphasized far beyond its due importance, I am, naturally, pleased that the "Final Report of the National Committee of Fifteen on Geometry Syllabus" allows me to continue in this opinion. This report which appears in the MATHEMATICS TEACHER for December, 1912, and may also be obtained from the United States Commissioner of Education, should be in the hands of every high school teacher of mathematics. Another very helpful pamphlet is "The Report on Entrance Requirements of a Committee of the Society for the Promotion of Engineering Education," which may be obtained, presumably, from the Secretary of that Society, Ithaca, N. Y. The formulation of the mathematical requirements in this report appeals to me as being the most sensible and suggestive that I have seen in print. I would also like to call attention to the Syllabus of Mathematics published by the same society.*

What knowledge and power should the high school boy receive from his study of geometry? He should be familiar with the definitions and properties of the ordinary forms of geometry and should be able to represent them by figures neatly drawn as well as to find their areas, perimeters and volumes with accuracy and certainty. He should know the demonstration of the principal theorems as given in some standard text-book and should be able to give brief and accurate original demonstrations of simple subsidiary theorems.

By omitting the unnecessary theorems and by restricting the number of originals, he will have time for an elementary course in solid geometry. To require solid geometry, is, I suppose, contrary to the movement of the times, but I still feel that the boy who is not going to college should have at least a brief look into

the geometry of space. Leave out the theory of limits, or, at any rate any attempt at proofs, also the incommensurable cases and use the time thus saved to gain knowledge that will really mean something to the boy. After he goes to college he will still find these subjects sufficiently difficult. On the other hand, it is well worth while to teach the high school boy the definition of sine and co-sine, tangent and cotangent and their use in mensuration.

Thus far has been set forth the writer's idea of what mathematical knowledge and ability should be expected of the pupil who is to *continue* mathematics in college. What about the pupil who "does not like" mathematics? Is there a place for him in college, and what mathematics should he be required to do in the preparatory school? The pupils who really cannot learn mathematics are rare; those who, with skillful handling, may not be led to enjoy the subject are comparatively few. All high school pupils should be "exposed" to a year of algebra and a year of plane geometry. If it does not "take," they should not for that reason be refused a high school diploma, nor, in rare cases, admission to college, if they present high scholarship in other lines.

I suppose that every college professor of mathematics has had some painful experiences with students entirely satisfactory in other subjects who have repeated freshman mathematics each succeeding year until graduated "by grace of the faculty." Such cases reconcile us to having mathematics placed among the elective subjects of the curriculum.

In conclusion, let me say that, in my opinion, mathematical teaching in the high school is, at least, as good as that in college; that only a small fraction of freshman failures in mathematics can properly be charged to the high school and that this small fraction may be made still smaller by placing the emphasis of preparation upon quality rather than quantity and by selecting and strongly emphasizing the most vital topics. The boy who knows a few things thoroughly and has really learned to think, and, perhaps, to enjoy thinking in mathematics, is more likely to succeed than he who has merely plodded through the whole text.

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FRESHMAN ALGEBRA AND THE AVERAGE FRESHMAN.

By Ella Durgin Gray.

Let us first consider our pupils in two classes, those for whom algebra has a practical value and those for whom it is merely a culture study. Now, as probably most of us agree, the practical value of algebra, in the sense of the man in the street, is almost negligible. Personally, apart from my professional connection with it, I can recall but one instance of its having been of service to me. If by practical value we mean its value to those who practise it, our first class is limited to those pupils who will use algebra in other courses, at high school or at college. Those of this class who elect review algebra in high school, usually find that their grade in the review has been largely determined during their freshman year by the degree of facility they acquired in interpreting and manipulating notations and by the clearness with which they distinguished certain principles and their applications. For the time allowed review algebra is generally adequate only for those who have already some insight and skill to test these in more complicated and lengthy exercises and to extend somewhat the ground previously covered. The advanced pupil who is hazy and uncertain about fundamental notions and procedures is himself at a great disadvantage and he retards greatly the progress of the others. The advanced pupil who has acquired a merely mechanical proficiency frequently has too large an estimate of his knowledge and therefore does not present so open a mind towards efforts to clarify, deepen and broaden his vision as he would in the discussion of new matter. To the college preparatory student, then, his freshman algebra is highly important.

While by far the majority of our pupils are not among those for whom algebra has a practical value, yet with present opportunities in evening, part-time, university extension, and correspondence courses, in mathematics, pure and applied, one never can be sure that any pupil may not later have practical need of

algebra in the sense that has been defined. Every pupil, then, has a potential claim to that same grasp of the subject matter that we owe the college preparatory student.

However, for those not professedly in the first class, algebra is made a required study on account of the mental training it affords. What is the nature of this mental training? An obdurate and tiny boy once, in an after session conference, reiterated, to all my efforts to enlist his interest in his work, "I ain't got no use for algebra," and I was moved to ask myself, "Has he? What is the use of algebra to those in the second class? Would I, for instance, want some little girl for whom I cared to give her time to it daily unless it was to yield her other than such returns as powers of accuracy and persistence which she could gain from other subjects, richer to her in content; particularly, if the study was to mean to her daily toil, with the contracted muscles that are the reflex from a perturbed spirit?" And considering these questions, I have felt convinced that algebra affords a pleasurable medium wherein the average pupil may acquire, in greater or less degree, certain of those mental powers that in exceptional pupils have already developed, naturally or through training, and that these powers, reacting directly upon the pupil's comprehension of the subject matter, make possible greater practical value to him who needs it. Each exercise in algebra—each example, so called—constitutes for the pupil a unit of achievement and here, as in many games, the satisfaction of attainment is both an appeal to interest and a motive force. In making it possible for a pupil to score well in these units, we transform him from a listless taskman to one who exercises his activities the more because of satisfaction through doing so.

By reviewing some of these respects in which algebra affords opportunity for mental development, we may, perhaps, agree that for the advantage of each class of pupils, those for whom algebra has practical value and those for whom it is merely cultural, the same subject matter, the same class work, gives each his due. If the development of mental power among those in the first class enables them to make the content of greater practical value, and if the content of algebra furnishes the freshmen of the second class with the handiest substance available to them for sharpening their infant mental incisors, then the distinction made

between these two classes results in our clarifying our own understanding of what is for the good of each.

What, then, are the grounds in which algebra is fertile for mental growth? One, in which it sometimes does not yield all it might, is that for cultivating the power of discrimination between the accidental and the characteristic. How frequently our pupils appear to understand a principle, applying it with some accuracy when first it is presented to them, and while the exercise are all of a kind, only to become much entangled later when it occurs in a miscellaneous set, or in a complicated or a literal expression. They do not see beneath the form, nor always all of the form. A marked instance of this failure to discriminate occurs when the class reaches the chapter on factoring. The average pupil recites glibly as case after case is taken up, but when confronted with a review set, he confuses the various types, lamentably. If $9a^2c - c^5$ occurs among some examples of the binomial that is the difference of two squares, he is likely to give as the factors of $9a^2c - c^5$, some binomial difference and some binomial sum. Whether $9a^2c - c^5$ is itself a difference of two squares and, if so, what are the roots of these squares, are not matters of interest or inquiry. The number of terms and the arangements of signs, both in the product and in the factors, appear to have been for him the salient features in the type of the difference of two squares. It is possible here with gain to him in the double sense before noted, that of hold on subject matter and of mental power, to set before him, at the time of his first acquaintance with the chapter on factoring, all the main types simultaneously and to force him from the start to discriminate among them. For about five years I have tried this with very satisfactory concrete results and in the belief that this method is more educational. It needs to be reinforced by presenting these types both in formulas and in precisely worded descriptions and with both simple and fairly difficult illustrations and by training the pupils to prove their factoring by mental multiplication and thus find their errors as they make them, and it needs to be preceded by a preparatory exercise in which is another opportunity for discrimination. Many pupils have only vague notions as to what are squares and what are cubes. After they have formed tables of the squares and cubes of roots, say

from $2a^2$ to $10a^{10}$, they can profitably drill in naming as squares, as cubes, as both, as neither, such expressions as 9, a^9, $9a^2$, $9a^2c$, a^6, 6, 1, $8a^9$, $9a^8$, $64a^6$, $64a^9$, etc. Even if the chapter on factoring was not an essential one, it would be valuable for the practice it may afford in observing and classifying characteristics.

Algebra furnishes an extensive field for exercise of judgment. A pupil working on simultaneous linear equations can readily be persuaded of the advantage of keeping his numerical coefficients as small as possible, and can become accustomed to decide before entering upon elimination, which elimination will involve the smaller multipliers. In literal simultaneous equations of the first degree, he has a chance for decision as to whether it will involve less labor to obtain the second unknown by substitution of the first or to get it as he got the first. In simplifying a complex fraction, it is frequently worth his while to make a choice between treating it as a quotient of polynomials, such as he may already have simplified in his study of division of fractions, and regarding it merely as a fraction, to be simplified by transforming both terms by means of a suitable multiplier. Again, in reducing certain complicated expressions in the chapter on the theory of exponents, he can be shown that it costs him time and effort not to consider whether he had better simplify through regarding each as a case of $(a^m)^n$, of $a^{m/n}$ or a^{-n}, etc. So many expressions in this chapter may be simplified by several methods. $a^{-1/2}b^{3/4}c$ times $a^{1/2}b^{-2}c^{-5}$ and $(a^{1/3})^{-5/4}$ are illustrations of kinds in which he will suffer by not exercising judgment as to method. The chapters on Theory of Exponents and on Radicals give especially favorable ground for training a pupil to find and maintain a mental equilibrium. In this connection it may be noted that frequent drill on the fact that while $(ab)^m = a^m b^m$, on the other hand $(a+b)^m$ does not equal $a^m + b^m$ would prevent a large proportion of errors made in radical expressions. The average pupil, in these and in other instances, does not use his judgment however unless he has been trained to do so. Left to his own devices, his procedures are most haphazard. In other subjects, his judgment is exercised, to be sure, but in algebra he can experience, definitely and directly, the effect of failure to use it.

From algebra may be developed practice in generalization. In

factoring, for instance, the pupil who clearly sees that an expression such as $(a-b)^2-(c-d)^2$ is essentially the same as his type form x^2-y^2 and who realizes that his factoring is accomplished when he has expressed $(a-b)^2-(c-d)^2$ as a binomial times a binomial, and that whatever else he does is in the way of simplifying those factors, and who has, furthermore, learned to group certain three, four and six term polynomials so that they present the same type, that pupil carries in his mind one kind of factoring instead of many, and he has had his perceptions quickened by his having here discerned the type among individuals. It is true that he can be trained to give with accuracy the simplified forms of the factors without so much ado. That is, given $(a-b)^2-(c-d)^2$, he may be trained to state directly the four term factors $(a-b-c+d)$ $(a-b+c-d)$ but the chances are that he gets them by some such mental path as this: "They always come out of the first parenthesis with the same signs and out of the second parenthesis, once with the same and once with the opposite signs." Of course, if the concrete result, the answer so called, is the only thing of importance, no criticism of that method is in order, provided, of course, that he is sure to remember it and provided, also, that there is no objection to lumbering his mind with two sticks when one stick would suffice. The point I am trying to make here is that whatever a pupil may do mentally, he should be able to supply the successive stages by which it is properly developed, if challenged.

Again, algebra may give development in that flexibility of mind which comes in some degree from practice in expressing the same thought in different media. Problems give an opportunity for this. The majority of pupils are studying a foreign language and they can readily be brought to regard as a vocabulary of words and phrases what is usually called the statement of the problem and to consider the equation itself as a translation from English into algebraic language. The equation can then be regarded as the expression of a thought, and not as an arrangement of x's and numbers, an arrangement chanced upon frequently after the rejection of others that refused to work out into satisfactory results, or imitated slavishly from a model, an arrangement of uncertain meaning but acceptable because from it can be obtained some desired numbers. Pupils not taking a

foreign language can without much difficulty understand and adopt such a treatment of problems. The habit of before solution translating the equation back into literal English and of noting whether this literal rendering adequately expresses the problem, checks errors and tends towards further flexibility in expression.

Algebra may be so taught as to stimulate purposive thinking. Given $s = [n(a + l)]/2$, to solve for l, pupils can be trained to see that to release l from the undesired relations that hold it, they must do as they would do in the world of affairs. If they would unlock what has been locked, they must reverse the operation of locking, if they would unscrew what has been screwed, they must reverse the operation of screwing, so, to eliminate 2 from the fraction $[n(a + l)]/2$ they must determine that 2 is held there by division and must remove it by the inverse of division, multiplying it away and then, to preserve the equation, multiplying each other term, also by two. Similar purposive thinking, sufficiently continued, gives him l in the terms of the other letters. Such mental procedure is not difficult for even the average freshman mind. Training for it can begin when first he meets the equation. Accustomed to such purposive attacks as this, should the algebra of his physics so confound him as it is sometimes reported to do?

Again, it being human to wish to use and to extend a power that is felt, algebra, in so continuously setting before a pupil units of achievement, creates in him a pleasure, more or less consciously felt, in the possession of the power of achieving and fosters persistency and patience through his exercise of this power.

If the foregoing are some of our aims, shall we adapt our teaching so as to realize more of them in the average freshman? What is the average freshman? If he is the type numerically conspicuous in our classes, his most striking characteristic is his tendency to be impressed by what appears rather than by what is. He is a being with a facility (sometimes fatal to him when we do not find him out) for seeming to know what he does not know. Enchain his attention by an explanation or give him a model to follow and he will show skill in those exercises that are superficially as well as actually similar. Then direct him to some

that involve the same principles but look different, such as literal forms, and he is quite likely to say he "cannot do that kind," or to do them, to his own undoing. Listen to him, the average Freshman in his natural state, as he explains to a classmate the procedure in such an equation as $x + b = c$. "You take b," he says, "and change its sign and put it on the other side"; or hear him tell another what to do in the equation $x/3 + y/2 = 5$. "You take this denominator," he says, "and multiply that numerator and take that denominator and multiply this numerator and multiply the whole number by both of 'em." It is true that we ourselves manipulate mechanically thus and find it restful to do so but along with the apparent process our minds have long since connoted the real one; hence we are guarded from deducing, as they do, incorrect procedures from mechanical practices.

The average freshman is impatient of detail in writing out his work and is prone to attempt to do mentally more than his mastery of technique warrants. While solving equations, for instance, he makes more transformations in one writing than he is capable of making accurately. Frequently, the time lost in this way, first, that used in calculating the results and, secondly, that in tracing errors when the completed work reveals that calculations were not correct, about equals the time it would take to write rapidly with little mental strain the successive transformations, one by one. He has gained nothing in time and has lost, probably, in courage. "Let your fingers be the servants of your mind" is a precept he may profitably heed throughout his freshman year. Rapidity and accuracy in transformation become possible when he proceeds one step at a time and the established habit furnishes him with drill on elements.

He has no sense of form in written work but is slipshod and fragmentary. For instance, he links equations together by equality signs and he is likely to write down the particular part of an expression that is engaging his attention and link it by an equality sign, at one or at both ends, to the whole expression—that is, he may write $16a^4 + 16a^3 + 4a^2 = 4a^2(4a^2 + 4a + 1) = (2a + 1)^2$ and $(a - b)/2 + (a + b)/3 = 3(a - b) + 2(a + b) = (5a - b)/6$.

He does not correllate. Algebra is to him a diverse multiplicity. He thinks of a certain section of the book, for instance,

as "P'rentheses." He has to be told that here he is performing the same operations that he has been performing, the only distinctions being that he is translating his directions from a notation instead of seeing them in English, and is writing his results in horizontal lines. He does not reflect that an argument against thinking that $a+b$ equals $\sqrt{a^2+b^2}$ lies in the fact that $a+b$ squares into a trinomial. He does not notice that, given a new type of an equation, his method is to obtain from it one of the types already familiar; that from a linear equation of one unknown involving parentheses, he seeks, generally, one of the type $5x-2=2x+7$; that given a pair of equations in two unknowns, his concern should be to get from them one in one unknown; that from a set of three in three unknowns, he is to obtain, generally, two in two unknowns; that from a fractional equation he is to get an integral equation, and, from a quadratic, a linear.

To him, simplifying the radical $\sqrt{2/3}$ is an altogether distinct process from simplifying the radical $\sqrt{2/9}$. He has to be made to see that after the transformation of $\sqrt{2/3}$ to the $\sqrt{6/9}$ he merely has his previous case. In short, left to himself, he will tuck each captive idea into its own separate brain cell.

Finally, and in this we find our encouragement and his hope, the average freshman is a being susceptible to a considerable degree to training in clear and unified thinking. In the attempts to so train him, we may lay ourselves liable to the charge of making his work too easy for him. Our reply to such criticism may spring from the conviction that whatever we accomplish in the way of unifying and clarifying his ideas and in obtaining precision in his exercises, we cannot have enervated him if we have given him the freedom and the desire to use more mental powers than he otherwise would use.

What of our presentation to him of the subject matter? Are our sequences advantageous to him? We are under no necessity to keep the pace and the route of our particular text-book since many of the exercises in algebra can be taken out of the order in which they are set. Wherever it is possible by a change in arrangement to gain in time or in economy of the pupils' effort, that change is worth making. You need leisure at certain stages in teaching algebra and you can secure it in this way.

For instance, some books introduce multiplication by the case of a monomial times a monomial, following this by monomial times polynomial, before giving polynomial times polynomial, making two lessons prior to the last case. Those two lessons are not needed. A few minutes' oral drill on monomial times monomial, a few moments' attention to the likenesses and dissimilarities between the methods of operation in arithmetic and in algebra and the class can be launched successfully upon multiplying polynomial by polynomial. The case of adding a whole number and a fraction needs no individual treatment but can be brought under the addition of fractions by regarding the whole number as the numerator of a fraction whose denominator is one, the addition thus becoming that of fractions. Literal equations need not be considered by themselves. In taking up new types of equations, we can assign in each lesson a few literal ones along with the numerical. Addition need not be given the complicated consideration it receives in some books. One of the recent admirable text-books sets it forth under three captions: To add two or more positive numbers; To add two or more negative numbers; To add a positive and a negative number. Now a pupil can easily manage all these without difficulty—as one case, by being told to add positive and negative numerals as he would find a man's financial condition from his liabilities and his assets. If he questions, "When you add in algebra, do you sometimes subtract and sometimes add," he is satisfied by the reply, "When you add numerals in algebra, you combine them as you would a man's debts and resources." The phrase "algebraic sum" is found in many text-books. Is it not conducive to a pupil's thinking that adding in arithmetic is essentially different from adding numerals in algebra?

Multiplication can to advantage be taught directly after addition. Before the mind is confused by subtraction, the inverse of addition, and with very little expenditure of time or effort in teaching multiplication, further drill in addition can be found in summing the partial products. The interpretation of notations can join with further drill on multiplication by the introduction at this point to parenthetical expressions involving only multiplication. If any one questions, as I have several times been asked, how explain that minus times minus equals plus

without an understanding of subtraction, he may be referred to J. W. Young's "Fundamental Concepts of Algebra and Geometry" and to Fine's "The Number System of Algebra." A five minute game, minus times plus, plus times minus, minus times minus, times plus, times plus, times—minus, minus times—minus, minus times minus, minus times plus, etc., fixes the law satisfactorily for them. Also, multiplication and division of fractions can follow reduction of fractions to lowest terms, since they involve so little more teaching and furnish so abundant practice in reduction. By the way, why do we allow pupils to follows book models in the multiplication of fractions, rewriting each fraction with its numerator and denominator in prime factor form and then cancelling criss cross from one numerator to another denominator, multiplying finally the remnants of the fractions. It is an unsightly method and it does not accord with any rules or principles they have had. They are not formally multiplying the given fractions and they are not formally reducing any single fraction to its lowest terms. That principles might be furnished them which would cover such a procedure seems beside the question. If we are training them for any thing but concrete results why not hold them to a regard for what is within their knowledge. By multiplication of the fractions directly into one fraction, expressing numerator and denominator of the product in their factored forms, and making a reduction of that fraction to lowest terms they act within their rules and use a more compact form. Purposive thinking in this connection may raise to them this question, "Why multiply numerator and denominator under a factorial rather than under a simplified form?"

After long division has furnished a drill on subtraction, any lingering confusion between the processes of addition and subtraction can be clarified in simultaneous linear equations, in two and in three unknowns, the class thus gaining considerable technique before going further. The graph and the indeterminate equation introduced at this point, in checking one another and in giving drill on the substitution of negative numbers, furnish them with easily determined tests of their own accuracy. Besides, if pairs of equations are now given to be solved simultaneously, indeterminately, graphically, the pupil gets a drill on

passing rapidly through a succession of different operations when each operation is one in which he is fairly accurate. Without this, unprepared for much diversity, he enters shortly upon addition of fractions where the rapid passing through a variety of processes, some not yet sufficiently drilled, to wit; recognizing factors, determining the least common multiple of denominators, reduction to equivalent fractions having their least common denominator, addition into a single fraction, simplification of the numerator, and reduction to lowest terms, all distract his attention from accuracy.

Such changes of order as these result in furnishing some of the material for drill which text-books lack. Mathematical teaching as a rule compares unfavorably with that of music and languages in the matter of drill on elements.

We can clarify some ideas for the average pupil by refusing to permit in the class room words that connote what is mechanical rather than mathematical. Cancel and transpose are two such terms. The average pupil delights in cancelling. There is something in the bold sweep of the arm as he dashes out this, out that, something in this summary rejection of what he would eliminate, that endears it to him. But what does he mean when he says he cancels? When he has $x^2 + 6x = x^2 + 8$ he says x^2 cancels. Here he should mean that he adds $-x^2$ to each member of his equation. He says when he has $[x^2(a+b)]/[x^2(a-b)]$ that x^2 cancels Here he should mean that he divides numerator and denominator both by x^2. As a matter of fact all he does mean is that in each case x^2 occurs in two places and that he is pleased to rid himself of it by crossing it out. Small wonder that soon he crossed out x in $(b+x)/ax$. If he is not quite so reckless as this, he is a very unusual pupil if at some time he does not make the same sort of an error in a more complicated expression. Such mistakes are common among advanced pupils of some attainment. We can minimize their frequency by explicitly instead of tacitly teaching that the removal of a factor from a product is a method of division by that factor, by the discussion of possible transformations of a fraction and by seeing to it, through frequent challenge and cross question, that, when they reduce fractions to lowest terms, they are intelligently following the rule which they

are taught; namely, "divide numerator and denominator by their highest common factor." Transpose is another misleading term, accountable for thoughtless work. For them it means, change the sign and move. Let me illustrate from the work of a boy who first appeared to me in the middle of his junior review. He is faithful, attentive, fairly accurate and intelligent. In one of his previous tests I had noticed badly mangled equational forms and had criticized them. This, therefore, seems to have resulted from no momentary aberration, such as even the best of pupils may experience. The equation was $x - 3y = 5$, and his expression for y, $y = 8 - x$. His mind must have worked thus: $x - 3y = 5$, x is in the way on the left side so change its sign and put it on the right, getting $-3y = 5 - x$. -3 is also in the way on the life side, so also change its sign and put it on the right. $-3y = 5 - x$, $y = 5 - x + 3$ and $y = 8 - x$.

We can clarify ideas by letting the mind derive them from notations rather than from a variety of rules. Is not addition in algebra a matter of notation and is not much that goes under the name of addition, the simplifying of a sum by means chiefly of monomial factoring, through, of course, the distributive, associative and commutative laws. We add $6a$ and $-2b$ by writing $6a - 2b$. We can add $6a$ and $-2a$ by writing $6a - 2a$. But because the latter can be rewritten $(6 - 2)a$ and then as $4a$, we get a simpler form. It may not seem wise at the beginning of algebra to teach additions of polynomials thus subtly, but when addition of fractions is reached, a brief explanation of what underlies the addition of polynomials can introduce the class to such an understanding of the method for the addition of fractions. If he is first drilled a little on the fact that a fraction may factor into its numerator and the reciprocal of its denominator, he can see why he reduces his fractions to equivalent fractions having their least common denominator, he can then factor his polynomial into the reciprocal of the least common denominator and the resulting quotient. This resulting quotient, if he has been well drilled on simplifying expressions containing parentheses involving both subtractions and multiplications, he can readily set down as a factor free from compound terms; then, if he has already had multiplication of fractions and if he

regards the quotient factor above referred to, since it is an integer, as the numerator of a fraction whose denominator is unity, the rest of the simplification becomes the multiplication of two fractions. My experience with this method of adding fractions has been most satisfactory. Pupils take hold of it readily from the first. When they take up transformation of fractional to integral equations, they no longer are in danger of confusing the two, and consequently of dropping off denominators when they add fractions and their work on complex fractions by the methods involving addition is very straightforward and clear. Of course in complicated expressions, such a method, because of the parentheses it involves, might seem to be something of a nuisance but by the time pupils are able to deal with such complicated expressions they can perform mentally that step in which this method differs from the conventional one, namely the step where the factoring is done, leaving the form of the work the conventional form. And the addition of radicals, by this same method of factoring, becomes the simplification of radical terms of a polynomial sum, and the substitution of a factorial form for every group having a common surd factor. Such a method does not involve, as does that in many books, taking the polynomial to pieces and later putting it together again, frequently with some injury of signs. Much of the difficulty of highest common factor and least common multiple can be removed by insistence on interpretation of these words themselves. No rule is necessary for the highest common factor if the pupil can be brought to see that he needs only to determine what are the factors that are common to all the expressions. The word common, by the way, usually is without significance to him. Its meaning in this connection can be lodged in his mind by discussing its application in the term, Boston common. The least common multiple can be handled without intervention of a rule if the pupils can be made to see, by arithmetical and algebraic illustrations, first, what a multiple is, and secondly, that a multiple of an expression must be a multiple of every factor of that expression.

All he needs to know about exponents, before he comes to the chapter on their theory, is contained in the statement $a^2a^3 = aa\ aaa = a^5$. When he asks, "Do you add the exponents when you

multiply a^3 and a^4," a sufficient reply is, "You know $a^2a^3 = aa\ aaa = a^5$, what do you think a^3a^4 can mean?" As he has, in other ways, much practice in addition, why distract his mind by mechanical rules, almost necessary if we introduce him to literal integral exponents before we come to the chapter on the theory of exponents. When he reaches this chapter, eight statements furnish him with a sufficient working basis for his needs. These are,

as $\qquad a^2a^3 = aa\ aaa = a^5$, so $a^m a^n = a^{m+n}$

as $\qquad (a^2)^3 = aa\ aa\ aa = a^6$, so $(a^m)^n = a^{mn}$

as $\qquad (a/b)^2 = a/b \times a/b = a^2/b^2$, so $(a/b)^m = a^m/b^m$

as $\qquad (ab)^2 = ab\ ab = a^2b^2$, so $(ab)^m = a^m b^m$.

If $a^{2/3}a^{2/3}a^{2/3} = a^2$, then $a^{2/3} = \sqrt[3]{a^2}$, and similarly, $a^{m/n} = \sqrt[n]{a^m}$.

If also $a^{m/n} = (a^{1/n})^m$, then $a^{m/n} = (\sqrt[n]{a})^m$.

If $\qquad a^0 a^m = a^m$, then $a^0 = 1$.

If $\qquad a^{-m}a^m = 1$, then $a^{-m} = 1/a^m$ and $a^m = 1/a^{-m}$.

His accuracy can be increased by providing him with checks upon his possible errors. There are many checks besides that by numerical substitution. It is a question to what extent the weaknesses of a pupil in arithmetic should discourage his progress in algebra. When a numerical check fails to verify his answer, the error frequently is not in the algebraic work. If the desire is to use the freshman algebra to gain power in numerical computation, much practice of this sort is worth while. But for gaining technique in algebra there are other checks more valuable, in that they locate the error and in that the check itself is not so likely to be erroneous. For instance, errors in subtractions may be checked by mental additions of the remainder to the subtrahend. Classes trained to this habit will not present long division examples perforated with many errors in subtraction and yet, in their parlance, "coming out even." The correctness of signs in an expression put into a negative parenthesis may be tested by a mental removal of the expression from the parenthesis. Multiplication after each factoring checks errors there. This is of particular advantage when the factoring is first taught. If the practice is insisted on, the pupil

will at once have means of determining what types he has or has not understood and where he is or is not careless in using them. When fractions have been reduced to equivalent fractions having their least common denominator, the mental reduction of each fraction to its lowest terms and the comparison of the result with the original fractions show up mistakes when they occur. In solving problems, the habit of translating the equation back into English to see if a literal rendering is the equivalent of what was given in the problem, guards against the frittering away of time in solving useless equations. In finding highest common factors and least common multiples, the challenging of the results by certain queries can bring common sense into the lists. Some such queries are: Is each of these factors common to all the given expressions? Does this, which I am calling the highest common factor, contain all the factors common to all the given expression? Is this a multiple of each factor of each of the given expressions? Does it contain an unnecessary factor?

Algebra differs from most other subjects, excepting the languages, in that it is cumulative, so that if a pupil once gets below the class average he is at a greater disadvantage because of this. There is therefore this added reason for making it worth while to consider as many ways as possible for keeping a pupil in the game. When he has no confidence in his own accuracy, no means of testing it, when he has no way of judging whether to attribute failure to lack of understanding or to lack of care, what wonder that he becomes discouraged when he is entangled in his mistakes. Unsuccessful mental effort is far more fatiguing than successful mental effort. The more possible that accurate manipulation is made to the average freshman, by drill on elements and by training him to gauge his own results, the more courage and interest he will have for his daily encounter with his lesson. Besides, we are learning to use prophylactic measures against disease and crime. Why not use them in pedagogy?

My statements regarding the natural tendencies of the average freshman have been derived during the last fifteen years from interested and analytical reading of the test papers of some hundreds over a thousand freshman pupils and of many boys and

girls whom I have first met in review algebra. Each departure that I have advocated from the conventional book method I have worked out in the class room at least several times and have found practicable for freshmen. My deep convictions are that, in the same course both classes of our pupils may contentedly satisfy their individual needs, the one for the content of algebra, the other for the training experienced through it, and that the average freshman may participate to a considerable degree in clear and unified thinking.

SOMERVILLE, MASSACHUSETTS, HIGH SCHOOL.

NEW BOOKS.

Beyond the Atom. By JOHN COX. Cambridge: The University Press. G. P. Putnam's Sons American Representatives. Pp. 151. 40 cents net.

This essay is an attempt to tell in short compass the story of the discoveries which within the last decade have led beyond the atom. It is not so technical but that a student with a knowledge of elementary physics can read with profit and interest.

A General Course of Pure Mathematics. By ARTHUR L. BOWLEY. Oxford: The Clarendon Press. Pp. 272. $——.

Beginning with indices and ending with solid analytical geometry the author has in one book covered in a scholarly manner the principal body of the pure mathematics a student will want for the pass examination in the universities. It comprises algebra after quadratics, similar figures and projection in one plane in geometry, trigonometry, plane coördinate geometry, calculus and solid analytical geometry.

The Mechanistic Principle and the Non-Mechanical. By PAUL CARUS. Chicago: The Open Court Publishing Company. Pp. 125. $1.00.

This is an inquiry into fundamentals with extracts from representatives from both sides in which the author tries to harmonize the two.

The Principle of Relativity in the Light of the Philosophy of Science. By PAUL CARUS. Chicago: The Open Court Publishing Company. Pp. 105. $1.00.

In this book the author endeavors to show that the principle of relativity is not the new one it seems to be, but that as early as 1726 Bradley discovered that the fixed stars possessed a definite and peculiar motion of their own which was due to the motion of the earth about the sun and dependent on the time it takes light to reach the earth. He believes that a consideration of the history of its origin will clear the theory of much of its present mysticism.

Theory of Functions of a Complex Variable. By HEINRICH BURKHARDT. Translated from the Fourth German Edition by S. E. RESOR. Boston: D. C. Heath and Co. Pp. 432. $——.

Up to the publication of this volume there was no introductory book on this subject in English adapted to beginners which treated it from the twofold viewpoint of Weierstrass and Riemann. The translator and publishers have, therefore, done a good service to American students in putting out this excellent work of Burkhardt in English.

Constructive Text-Book of Practical Mathematics. By HORACE W. MARSH. Vol. II, Technical Algebra, Part I. New York: John Wiley & Sons. Pp. 428. $2.00 net.

This text is the result of an attempt on the part of the author to solve the problem of the teaching of mathematics, not from the point of view of the mathematician, but from the necessities of the student and the demands of the environment in which he is and into which he is to go.

Durell's Algebra, Book One. By FLETCHER DURELL. New York: Charles E. Merrill Company. Pp. 393. $1.00.

The authors main purpose in writing this book has been to simplify principles and give them interest. In extent it covers the work of the first year and so begins as to remove much of the initial difficulty usual to the subject. Effort is made to keep the interest alive by connecting the subject with everyday life.

Practical Mathematics. By NORMAN W. M'LACHLAN. London: Longmans, Green and Company. Pp. 184. 80 cents.

This book is designed for students in evening and day technical schools and covers geometry, algebra, trigonometry, including the use of logarithms.

Analytic Geometry and Principles of Algebra. By ALEXANDER ZIWET and LOUIS ALLEN HOPKINS. New York: The Macmillan Co. Pp. 369. $1.60.

The authors of this work have combined certain topics of algebra with analytic geometry, believing that in this way the student will get a better idea of the meaning and usefulness of the topics. Simultaneous linear equations are taken in connection with the straight line, quadratics in connection with the circle and something of numerical algebraic equations in general in connection with plotting.

A chapter is devoted to higher plane curves and several to solid analytic geometry.

A History of Japanese Mathematics. By DAVID EUGENE SMITH and YOSHIO MIKAMI. Chicago: The Open Court Publishing Company. Pp. 288. $3.00.

It is only in recent years that Japanese mathematics and mathematicians have attracted any attention in the western world. It is therefore very appropriate that at this time two men well qualified for the task should give us a history of Japanese mathematics. It starts back with the earliest period and comes down to quite recent times and makes very interesting reading.

Miss Billy Married. By ELEANOR H. PORTER. Boston: The Page Company. Pp. 383. $1.25.

The doctrine of joy and gladness is one which the world is in no

danger of having too much of. The present volume by the author of the *Glad Book* gives an account of how the happiness of one life radiated out to others and increased their joy and usefulness.

Alma's Junior Year. By LOUISE M. BREITENBACK. Boston: The Page Company. Pp. 376. $1.50.

This is the third volume in the Hadley Hall Series and shows Alma in a new capacity. She now has different problems to solve and new associates, but possesses the same charm and friendliness as when a year younger and a sophomore.

Essentials of Business Arithmetic. By GEORGE H. VAN TUYL. New York: American Book Company. Pp. 272. 70 cents.

Trigonometry. By FREDERICK ANDEREGG and EDWARD DRAKE ROE, JR. Boston: Ginn and Company. Pp. 108. 75 cents.

In this revised edition some demonstrations, which experience has proved too technical for the average elementary student to understand easily, are simplified without loss of vigor or generality; many examples which illustrate the practical side of trigonometry are added; and model solutions for the various cases of the right and oblique plane triangle are given. The book represents an effort to give as much trigonometry as is needed in an elementary course, with possibly the addition of a few examples to arouse the interest and test the ability of the stronger students, and is intended for the use of seniors in high schools and freshmen in colleges.

The 'Wellcome' Photographic Exposure Record and Diary for 1914. This excellent record is well worth the price (50 cents) asked for it. There is a special edition for the United States.

Diary and Time-Saver for 1914, published by Laird and Lee, of Chicago, is up to its usual standard of excellence and costs but 25 cents.

American Ideals, Character and Life. By HAMILTON WRIGHT MABIE. New York: The Macmillan Company. Pp. 341. $1.50 net.

The author of this book was our exchange lecturer to Japan and for the most part the addresses given were those delivered over there on the development of the American people. From its sane position it should do much in giving a more accurate view of the American spirit than that usually held.

Syllabus of Plane Geometry. Arranged in Groups for Emphasis and Method. By ROBERT R. GAFF, B.M.C. Durfee High School, Fall River, Mass. Price 25 cents.

This syllabus is an attempt to organize the subject of plane geometry under those important propositions which may be considered the founda-

tion proposition of each group. Where there is no proposition under which to organize, the theorems are grouped by classes, as those on congruence, on inequality of sects or angles, etc. The propositions are summarized progressively, the pupil being kept in remembrance of all the ways of accomplishing any desired purpose.

The syllabus puts into the teaching of plane geometry that group idea which was a feature of the report of the National Committee of Fifteen. It is a very interesting variation from the usual order, and has many points in its favor, the chief of which is, of course, the immediate correlating of propositions with a like purpose. Whether this correlating needs to be done in the first taking up of the course, or can equally well be done after several groups have been studied in some development order which includes no attempt to keep the different kinds separate, is an open question.

In some parts the loss of continuity that arises by taking parts of one discussion, such as the proofs that equal central angles subtend equal arcs, and that the greater of two central angles subtends the greater arc, under different heads at different times, seems greater than any gain that can result. In other parts, the definite summing up of subordinate propositions under one foundation theorem, as all theorems concerning the measurement of angles under the theorem on the measurement of a central angle, is well worth while, and although not strictly a deviation from regular practice, is at least an increase in emphasis.

Any teacher of geometry who wishes to keep in touch with the newer developments and to add to his teaching equipment, will find many interesting features in this book.

Archimedes' Werke. Von Sir Thomas L. Heath. Deutsch von Dr. Fritz Kliem. Berlin: Verlag von O. Häring, 1914. xii + 478 Seiten.

This is a translation of the English edition of 1897, which was a parallel to the works of Apollonius by Heath. In Apollonius and Archimedes we discover, according to Chales, the source of the two streams of geometry, viz., geometry of position (in Apollonius) and metric geometry (in Archimedes), metric geometry of the kind which through Kepler, Cavalieri, Fermat, Leibniz and Newton gradually led to the complete infinitesimal calculus.

The discovery of the Constantinople manuscript of Archimedes in 1906 by Heiberg made changes and additions necessary. In particular this manuscript revealed the methods by which Archimedes arrived at his conclusions, which are concealed in his other works, only the finished proofs being stated.

The German translation incorporates all the new material and Heath himself has looked after the changes made necessary and has approved of the translator's work, which appears to be done conscientiously and carefully. Under the circumstances the reviewer would prefer the translation to the original edition. There is a bibliography and a table of contents but no index. For those who are interested in the antiquities of mathematics, this work will prove a veritable antiquarium.

NOTES AND NEWS.

ALLYN AND BACON have just issued a trigonometry by E. J. Wilczynski which possesses some novel features. It is divided into two parts, the first being devoted to the theoretical and numerical solution of triangles, and the second to the treatment of the functions of the general angle. Other features are, its heuristic method, the way obtuse angles are introduced, carefully selected examples, careful detail in connection with numerical work, wide applications, explanation and use of slide rule, historical notes. It will undoubtedly be tried out with a good deal of interest.

THE annual meeting of The Association of Mathematical Teachers in New England was held at the Massachusetts Institute of Technology on December 6, 1913. The following officers were elected: *President*—William B. Carpenter, Mechanic Arts High School, Boston; *Vice-President*—Prof. F. C. Ferry Williams College; *Secretary*—H. D. Gaylord, 104 Hemenway St., Boston; *Treasurer*—F. W. Gentleman, Mechanic Arts High School, Boston; *Council Members*—Prof. Eva Chandler, Wellesley College, Harry B. Marsh, Technical High School, Springfield, Mass.

The Council Members whose terms have not yet expired are: Prof. Frederick S. Woods, Mass. Inst. of Technology; Ernest G. Hapgood, Girls' Latin School, Boston; Miss Sarah J. Bullock, High School, Arlington, Mass.; Edwin A. Shaw, High School, Natic, Mass.

REPORT OF THE JOINT MEETING OF THE ASSOCIATION OF MATHEMATICAL TEACHERS IN NEW ENGLAND AND THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN THE MIDDLE STATES AND MARYLAND, HELD AT THE NORMAL COLLEGE OF THE CITY OF NEW YORK, SATURDAY, FEB. 28, 1914.

The morning session, presided over by Mr. E. R. Smith, was opened with an address of welcome by President Davis, of the Normal College.

Mr. E. R. Smith, of the Park School, Baltimore, made an in-

formal report on the meeting of superintendents in Richmond, Va., in which he stated that a strong tendency was evident to class mathematics as a subject of but little value in the curriculum. He urged that every help possible be given the different committees now making the study of the values of the different courses of mathematics.

Prof. W. R. Ransom, Tufts College, Mass., spoke on the "Mathematical Pessimist."

The discussion was opened by Mr. Webb, of the Newark Central High School.

The third speaker was Miss Mary Evans, William Penn High School for Girls, Philadelphia, Pa., on "How Should Secondary Mathematics for Girls Differ from that for Boys."

The discussion was opened by Miss Robertson, of the Normal College.

The following motion was made and passed:

"*Resolved,* That a committee of five be appointed by the chair to consider and report to the Society concerning the following questions: (*a*) What mathematics should be taught in the seventh and eighth years of the grammar school course? (*b*) Does the mathematical work of those two years belong properly to the grammar school or to the high school? (*c*) What would be a proper (general) course in high school mathematics under the six-six plan?"

The afternoon session was presided over by Mr. W. B. Carpenter.

Mr. A. Harry Wheeler, English High School, Worcester, Mass., gave an address on "Methods" in algebra and geometry, illustrated by stereopticon slides.

The last speaker was Prof. George C. Chambers, University of Pennsylvania, who presented "A Study of the Reliability of Test Questions."

The meeting adjourned at 4 p. m.

NEW MEMBERS.

JAHN, LORA DOLORES, 121 Park Lane, Trenton, N. J.
SCOBORIA, C. P., Polytechnic Preparatory School, Brooklyn, N. Y.
KLINE, G. ALFRED, 6147 Chestnut St., Philadelphia, Pa.
TALMADGE, MISS BESSIE J., Hammondsport, N. Y.
PAGE, DONALD TAYLOR, 215 W. 23d St., New York City.
LATHROP, HARRY J., 43 Chappel St., Brockport, N. Y.
MAYER, E. S., West Chester Military Academy, Peekskill, N. Y.
GARLOCK, MISS EDITH A., 329 Park Ave., Leonia, N. J.
CRAMBLET, W. H., University of Rochester, Rochester, N. Y.
FROST, A. E., University of Pittsburgh, Pittsburgh, Pa.
BURROWS, B. E., Saranac Lake, N. Y.
WHITE, HOMER O., 145 Walnut St., Corning, N. Y.
BARNES, EDGAR S., Normal School, Geneseo, N. Y.
SMITH, MISS MARY RUSSELL, Chatham, N. Y.
SCALES, MISS EDNA R., 828 Windsor Sq., Philadelphia, Pa.
COOPER, MISS ELIZABETH MORGAN, The Baldwin School, Bryn Mawr, Pa.
COUNTRYMAN, R. LEROY, Normal School, Geneseo, N. Y.
BECKER, MISS CAROLINE F., 420 N. Carey St., Baltimore, Md.

Benway, Miss Mabel Reed, 62 Pierrepont St., Brooklyn, N. Y.
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Bull, Miss Helen L., 19 Melbourne Ave., Mamaroneck, N. Y.
Burr, Miss Nellie, 310 E. 56th St., New York City.
Cook, Miss M. Imogene, 100 Park St., Montclair, N. J.
Egan, Miss Hannah M., 1848 Mayflower Ave., New York City.
Guiry, Miss Martha A., 990 Second Ave., New York City.
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